

# WHAT YOU DIDN'T WANT TO KNOW ABOUT CANONICAL ALGEBRAS

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**ABSTRACT.** Canonical algebras, introduced by C.M. Ringel in 1984, play an important role in the representation theory of finite dimensional algebras. They are equipped with a large contact surface to many further mathematical subjects like function theory, 3-manifolds, singularity theory, commutative algebra and algebraic geometry. We show in this paper that canonical algebras are characterized by a number of interesting extremal properties (among the class of endomorphism rings of tilting bundles on a weighted projective line). We also study the corresponding class of algebras antipodal to canonical ones. This sheds new light on an old question: Why are the canonical algebras canonical?

## INTRODUCTION

Canonical algebras were introduced by C. M. Ringel in 1984 [Rin84] in order to solve an intriguing problem concerning the representation type of a certain class of finite dimensional algebras, now called tubular. When introducing weighted projective lines in 1987 [GL87], Geigle and Lenzing showed that canonical algebras arise as endomorphism rings of naturally formed tilting bundles, consisting of line bundles. Due to this fact, canonical algebras own a large contact surface to other parts of mathematics, classical and modern.

Indeed, the *canonical relations*

$$(1) \quad x_i^{p_i} = x_2^{p_2} - \lambda_i x_1^{p_i}, \quad i = 3, \dots, t$$

defining the canonical algebras already appeared 1882, respectively 1884, in the work of H. Poincaré [Poi82, p. 237] (see also [Poi85, p. 183]) and F. Klein [Kle93] yielding a link to Fuchsian singularities, respectively Kleinian (i.e. simple) singularities. For modern accounts on this aspect we refer to work of J. Milnor [Mil75] and W. D. Neumann [Neu77], compare also [Len94].

An important feature of the canonical relations is the (graded) factoriality of the commutative  $k$ -algebra  $S(\underline{p}, \underline{\lambda}) = k[x_1, \dots, x_t]/I$ , where the ideal  $I$  is generated by the canonical relations (1). Moreover, by Mori [Mor77] and Kussin [Kus98] graded factoriality determines the algebras  $S(\underline{p}, \underline{\lambda})$  uniquely (among the affine algebras of Krull dimension two).

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2010 *Mathematics Subject Classification.* Primary: 16G20, , Secondary: 14H45.

*Key words and phrases.* canonical algebra, concealed canonical algebra, weighted projective line, Hübner reflection, tubular mutation.

The first named author was supported by DGAPA and PAPIIT, Universidad Nacional Autónoma de México. The second named author was supported by the Max Planck Institute for Mathematics, Bonn.

This paper presents another record of *extremal properties* characterizing canonical algebras, completing research by Ringel [Rin09] on the challenging question “*Why are the canonical algebras canonical?*”

The paper is organized as follows. Section 1 presents our main results on characterizations of canonical algebras in terms of maximality conditions. In Section 2 we recall those properties on weighted projective lines that are needed for their proofs in Section 3. Section 5 deals with concealed canonical algebras with properties antipodal to canonical ones. In Section 4 we show that characterizations of canonical algebras *within the class of tame concealed algebras* have a tendency not to generalize to the tubular or wild case.

As a general reference for weighted projective lines we refer to [GL87, LdlP97]. Concerning finite dimensional algebras and their representations, the monographs [SS07a, SS07b] and [Rin84] contain the relevant information.

## 1. EXTREMAL PROPERTIES OF CANONICAL ALGEBRAS

In this section we present the main results of our paper, all expressing a certain extremal property of canonical algebras. Let  $\mathbb{X} = \mathbb{X}(\underline{p}, \underline{\lambda})$  be a weighted projective line given by weight type  $\underline{p} = (p_1, \dots, p_t)$  and parameter sequence  $\underline{\lambda} = (\lambda_3, \dots, \lambda_t)$ . We denote by  $\mathbb{L}(\underline{p})$ , or just  $\mathbb{L}$ , the rank one abelian group generated by elements  $\vec{x}_1, \dots, \vec{x}_t$  subject to the relations  $p_1 \vec{x}_1 = \dots = p_t \vec{x}_t =: \vec{c}$ . Then  $T_{\text{can}}$ , the direct sum of all line bundles  $\mathcal{O}(\vec{x})$  with  $0 \leq \vec{x} \leq \vec{c}$ , is called the *canonical tilting bundle* on  $\mathbb{X}$ . Its endomorphism ring is the canonical algebra  $\Lambda = \Lambda(\underline{p}, \underline{\lambda})$  in the sense of Ringel [Rin84], given by the same data  $\underline{p}$  and  $\underline{\lambda}$ , see also Section 2.8. Throughout we denote by  $t = t(\mathbb{X})$ , or just  $t$ , the number of weights  $p_i \geq 2$  and by  $\bar{p} = \bar{p}(\mathbb{X})$ , or just  $\bar{p}$ , the least common multiple of  $p_1, \dots, p_t$ .

The complexity of the classification of indecomposables for  $\text{coh } \mathbb{X}$ , the category of coherent sheaves on  $\mathbb{X}$ , respectively for the category  $\text{mod } \Lambda$  of finite dimensional right  $\Lambda$ -modules, is determined by the (orbifold) Euler characteristic of  $\mathbb{X}$

$$\chi_{\mathbb{X}} = 2 - \sum_{i=1}^t \left(1 - \frac{1}{p_i}\right).$$

The representation type for both categories is tame domestic if  $\chi_{\mathbb{X}} > 0$ , tame tubular for  $\chi_{\mathbb{X}} = 0$  and wild for  $\chi_{\mathbb{X}} < 0$ , see [GL87] as well as [LR06, LM93] and [LdlP97] concerning more specific information.

For notations and definitions, otherwise, we refer to Section 2. We assume all tilting objects  $T = \oplus_{i=1}^n T_i$  on  $\mathbb{X}$  to be *multiplicity-free*, that is,  $T_1, \dots, T_n$  to be pairwise non-isomorphic. Throughout, we work over an algebraically closed base field  $k$  which is algebraically closed. If not stated otherwise, modules will always be right modules.

**Maximal number of line bundles.** The following result allows two different interpretations. First, it expresses unicity of the canonical tilting bundle if  $T$  attains the maximal possible number of line bundle summands. Second, it shows that the same assertion holds if we minimize the differences between ranks of the indecomposable summands of  $T$ . Note that the case of two weights behaves somewhat special because there exist tilting bundles, consisting of line bundles, whose endomorphism rings are not canonical.

**Theorem 1.1** (Maximal number of line bundles). *Let  $T$  be a tilting bundle on  $\mathbb{X}$  whose indecomposable summands all have the same rank  $r$ . Then  $r$  equals one. Moreover, assuming  $t(\mathbb{X}) \neq 2$ , then  $T$  is isomorphic to  $T_{\text{can}}$  up to a line bundle twist and, accordingly, the endomorphism ring of  $T$  is isomorphic to a canonical algebra.*

**Homogeneity.** We call a tilting sheaf  $T$  on  $\mathbb{X}$  *homogeneous* in  $\text{coh } \mathbb{X}$  (resp. in  $D^b(\text{coh } \mathbb{X})$ ) if for any two indecomposable summands  $T'$  and  $T''$  of  $T$  there exists a self-equivalence  $u$  of the abelian category  $\text{coh } \mathbb{X}$  (resp. of the triangulated category  $D^b(\text{coh } \mathbb{X})$ ) such that  $u$  maps  $T'$  to  $T''$ . By its very definition the canonical tilting bundle  $T_{\text{can}}$  is both homogeneous in  $\text{coh } \mathbb{X}$  and in  $D^b(\text{coh } \mathbb{X})$ . Our next theorem implies that, under mild restrictions, the canonical tilting bundle is characterized by a variety of homogeneity conditions.

**Theorem 1.2** (Homogeneity). *Let  $T$  be a tilting sheaf on  $\mathbb{X}$  with indecomposable summands  $T_1, \dots, T_n$ . Assume that one of the following conditions is satisfied.*

- (i)  *$T$  is homogeneous in  $\text{coh } \mathbb{X}$ .*
- (ii)  *$\mathbb{X}$  is not tubular, and  $T$  is homogeneous in  $D^b(\text{coh } \mathbb{X})$ .*
- (iii)  *$\mathbb{X}$  is not tubular, and the perpendicular categories  $T_i^\perp$ , formed in  $\text{coh } \mathbb{X}$  all have the same Coxeter polynomial.*
- (iv)  *$\mathbb{X}$  is not tubular, and the one-point extensions  $A[P_i]$  of  $A = \text{End}(T)$  with the  $i$ -th indecomposable projective  $A$ -module all have the same Coxeter polynomial.*

*Then all indecomposable summands of  $T$  have rank one and, assuming  $t(\mathbb{X}) \neq 2$ , the tilting bundle  $T$  is isomorphic to the canonical tilting bundle  $T_{\text{can}}$  up to a line bundle twist.*

In particular, condition (iii) (resp. (iv)) is satisfied if the categories  $T_i^\perp$  (resp. the algebras  $A[P_i]$ ) are pairwise derived equivalent.

Assuming  $\mathbb{X}$  tubular, we note that Example 4.1 presents a tilting bundle satisfying conditions (ii), (iii) and (iv) and whose endomorphism ring is not canonical.

**Maximal amount of bijections.** Assume  $T$  is a tilting bundle on  $\mathbb{X}$ , and  $A = \text{End}(T)$ . For  $\chi_{\mathbb{X}} \neq 0$  there exists a unique generic  $A$ -module. If  $\chi_{\mathbb{X}} = 0$ , that is, if  $\mathbb{X}$  and  $A$  are tubular, then there exists a rational family of generic  $A$ -modules, with one of them distinguished by  $T$ , and called the  *$T$ -distinguished generic  $A$ -module*, a name we also use for nonzero Euler characteristic. See Section 3.4 for references and the relevant definitions.

For the next result also compare [Rin09]. Note that the canonical configuration  $T_{\text{can}}$  always satisfies the condition stated below.

**Theorem 1.3** (Maximal amount of bijections). *Assume  $t(\mathbb{X}) \neq 2$ . Let  $T$  be a tilting bundle on  $\mathbb{X}$  with endomorphism ring  $A$ , and let  $G$  be the  $T$ -distinguished generic module. Then for each arrow  $\alpha : u \rightarrow v$  in the quiver of  $A$  the induced  $k$ -linear map  $G_\alpha : G_v \rightarrow G_u$  is injective or surjective. Moreover, each  $G_\alpha$  is bijective if and only if  $T = T_{\text{can}}$ , up to a line bundle twist.*

For  $t(\mathbb{X}) = 2$  each tilting bundle satisfies the above bijectivity condition. But there we will have tilting bundles with a non-canonical endomorphism ring.

**Maximal number of central simples.** If  $T$  is a tilting bundle on  $\mathbb{X}$  with endomorphism ring  $A$ , we identify  $\text{mod } A$  with the full subcategory of  $\text{D}^b(\text{coh } \mathbb{X})$ , consisting of all objects  $X$  satisfying  $\text{Hom}(T, X[n]) = 0$  for each integer  $n \neq 0$ . In particular, each simple  $A$ -module  $U$  belongs to  $\text{coh } \mathbb{X}$  or  $\text{vect } \mathbb{X}[1]$  where  $\text{vect } \mathbb{X}$  denotes the category of vector bundles on  $\mathbb{X}$ . Simple  $A$ -modules  $U$  belonging to  $\text{coh}_0 \mathbb{X}$ , the category of sheaves of finite length, are called *central*. Note, that this is equivalent for  $U$  to have rank zero.

**Theorem 1.4** (Maximal number of central simples). *Let  $T$  be a tilting bundle on  $\mathbb{X}$  with endomorphism ring  $A$ . Then the number of central simple  $A$ -modules is at most  $n - 2$ , where  $n$  is the rank of the Grothendieck group  $K_0(\text{coh } \mathbb{X})$ .*

*Moreover, the number of central simple  $A$ -modules equals  $n - 2$  if and only if  $T$  equals  $T_{\text{can}}$ , up to a line bundle twist.*

The question of the position of simple  $A$ -modules in the bounded derived category  $\text{D}^b(\text{coh } \mathbb{X})$  is also discussed in Section 5, see further [KS01] and [LS03, Section 5].

**Maximal width.** Let  $T$  be a tilting bundle on  $\mathbb{X}$ . We arrange its indecomposable direct summands  $T_i$ ,  $i = 1, \dots, n$ , such that their slopes satisfy  $\mu T_1 \leq \mu T_2 \leq \dots \leq \mu T_n$ . Then  $w(T) = \mu T_n - \mu T_1$  is called the *width* of  $T$ . Concerning slope and stability we refer to Section 2.5.

**Theorem 1.5** (Maximal width). *Let  $T$  be a tilting bundle on  $\mathbb{X}$  with endomorphism ring  $A$ . Then the width of  $T$  satisfies  $w(T) \leq \bar{p}(\mathbb{X})$ .*

*Conversely, assuming  $\chi_{\mathbb{X}} \geq 0$ , any tilting bundle  $T$  attaining the maximal width  $\bar{p}(\mathbb{X})$  equals  $T_{\text{can}}$  up to a line bundle twist.*

We expect that the theorem extends to negative Euler characteristic. In support of this, we mention the next proposition and point to experimental evidence obtained from examples constructed by means of Hübner reflections, compare Section 2.10.

**Proposition 1.6.** *Let  $T$  be a tilting bundle on  $\mathbb{X}$  with endomorphism ring  $A$ . We assume that  $T$  attains the maximal possible width  $\bar{p} = \bar{p}(\mathbb{X})$ . Then the following holds:*

- (i) *Each indecomposable direct summand of  $T$  of maximal (resp. minimal) slope is semistable.*
- (ii) *If there exist line bundles summands  $L$  and  $L'$  of  $T$  with  $\mu L' - \mu L = \bar{p}$  such that, moreover,  $L$  is a source, and  $L'$  is a sink of the quiver of  $A$ , then  $T = T_{\text{can}}$  up to a line bundle twist.*

**Extremality of the canonical relations.** Let  $R$  be a commutative, affine  $k$ -algebra, graded by an abelian group  $H$ . If  $x_1, \dots, x_n$  are homogeneous algebra generators of  $R$ , we always assume that their degrees generate  $H$ . We say that  $R$  is a *graded domain* if any product of non-zero homogeneous elements of  $R$  is non-zero. A non-zero homogeneous element  $\pi$  is called *prime* if  $R/R\pi$  is a graded domain. Finally, a graded domain  $R$  is called *graded factorial* if each non-zero homogeneous element of  $R$  is a finite product of homogeneous primes. Additionally, we always request that  $R_0 = k$  and that each homogeneous unit belongs to  $R_0$ .

Our next theorem expresses a strong unicity property of the canonical relations. Part (i) is [GL87, Prop. 1.3] while part (ii) is due to S. Mori in the  $\mathbb{Z}$ -graded case [Mor77] and to Kussin [Kus98] in general.

**Theorem 1.7.** (i) *Let  $\mathbb{X}$  be a weighted projective line. Then the  $\mathbb{L}$ -graded coordinate algebra  $S = k[x_1, \dots, x_t]/I$ , where  $I$  is the ideal generated by the canonical relations, is  $\mathbb{L}$ -graded factorial of Krull dimension two.*

(ii) *Assume, conversely, that  $R$  is an affine  $k$ -algebra of Krull-dimension two which is graded by an abelian group of rank one. If  $R$  is  $H$ -graded factorial then the graded algebras  $(R, H)$  and  $(S, \mathbb{L})$  are isomorphic, where  $S = S(\underline{p}, \underline{\lambda})$  for a suitable choice of  $\underline{p}$  and  $\underline{\lambda}$ .  $\square$*

We recall from [GL87] that the isomorphism classes of line bundles on a weighted projective line  $\mathbb{X}$  form a group with respect to the tensor product, called the *Picard group*  $\text{Pic } \mathbb{X}$  of  $\mathbb{X}$ . By means of the correspondence  $\vec{x} \mapsto \mathcal{O}(\vec{x})$  we may identify  $\mathbb{L}$  and  $\text{Pic } \mathbb{X}$ . The following corollary then states an important extremality property of the canonical relations.

**Corollary 1.8.** *Let  $R$  be an  $H$ -graded Cohen-Macaulay algebra which yields by sheafification (Serre construction) the category  $\text{coh } \mathbb{X}$  of coherent sheaves on a weighted projective line  $\mathbb{X}$ , and thus induces a monomorphism of groups  $j_R : H \hookrightarrow \text{Pic } \mathbb{X}$ ,  $h \mapsto R(h)$ . Then  $j_R$  is an isomorphism if and only if  $R$  is graded factorial, if and only if  $R$  is isomorphic to an algebra  $S(\underline{p}, \underline{\lambda})$  defined by canonical relations.  $\square$*

As mentioned in Section 2.8 the *squid*  $T_{\text{squid}}$  is competing with the canonical tilting bundle for the property of being the *most natural tilting sheaf*. The squid  $T_{\text{squid}}$ , compared to  $T_{\text{can}}$ , is accessible with less theoretical knowledge. The squid thus is easier to construct from general information about the category  $\text{coh } \mathbb{X}$ , compare [Len97b]. On the other hand, the squid does not contain any information on the *canonical relations*  $x_i^{p_i} = x_2^{p_2} - \lambda_i x_1^{p_1}$ ,  $i = 3, \dots, t$ , and thus lacks information vital for the link via the *projective coordinate algebra*  $S = S(\underline{p}, \underline{\lambda})$  to other branches of mathematics, among them commutative algebra, function theory, and singularity theory.

## 2. THE SET-UP

We recall that we work over an algebraically closed field  $k$ . For the convenience of the reader we collect relevant information about the *category of coherent sheaves*  $\text{coh } \mathbb{X}$  over a weighted projective line  $\mathbb{X}$ , see [GL87].

**2.1. The category of coherent sheaves.** The weighted projective line is given by a weight sequence  $\underline{p} = (p_1, \dots, p_t)$  with  $p_i \geq 2$  and a parameter sequence  $\underline{\lambda} = (\lambda_3, \dots, \lambda_t)$  of pairwise distinct, non-zero elements of the field  $k$ . We may further assume  $\lambda_3 = 1$ .

We recall that  $\mathbb{L}$  denotes the abelian group generated by elements  $\vec{x}_1, \dots, \vec{x}_t$  subject to the relations  $p_1 \vec{x}_1 = p_2 \vec{x}_2 = \dots = p_t \vec{x}_t =: \vec{c}$ . The element  $\vec{c}$  is called the *canonical element*. The *degree map* is the surjective homomorphism defined by

$$(2) \quad \delta: \mathbb{L} \rightarrow \mathbb{Z}, \quad \delta(\vec{x}_i) = \frac{\vec{p}}{p_i},$$

where  $\bar{p} = \text{lcm}\{p_1, \dots, p_t\}$ . The group  $\mathbb{L}$  has rank one with torsion subgroup  $\ker \delta$  and is partially ordered with positive cone  $\mathbb{L}_+ = \sum_{i=1}^t \mathbb{N}\vec{x}_i$ . This order is almost linear in the sense that for each  $\vec{x} \in \mathbb{L}$  we have the alternative

$$(3) \quad \text{either } \vec{x} \geq 0 \quad \text{or} \quad \vec{x} \leq \vec{c} + \vec{\omega}.$$

Here,  $\vec{\omega} = (t-2)\vec{c} - \sum_{i=1}^t \vec{x}_i$  is the *dualizing element* of  $\mathbb{L}$ .

The algebra  $S = k[x_1, \dots, x_t]/I$ , where  $I$  is the ideal generated by the *canonical relations*

$$x_i^{p_i} - (x_2^{p_2} - \lambda_i x_1^{p_1}), \quad i = 3, \dots, t$$

is  $\mathbb{L}$ -graded with  $x_i$  being homogeneous of degree  $\vec{x}_i$ , hence  $S = \bigoplus_{\vec{x} \in \mathbb{L}_+} S_{\vec{x}}$ . The group  $\mathbb{L}$  acts on the category  $\text{mod}^{\mathbb{L}} S$  of finitely generated  $\mathbb{L}$ -graded  $S$ -modules by *degree shift*  $M \mapsto M(\vec{x})$ .

The category of coherent sheaves on  $\mathbb{X}$  is obtained from  $S$  by Serre construction (=sheafification), compare [Ser55],

$$\text{coh } \mathbb{X} = \text{mod}^{\mathbb{L}} S / \text{mod}_0^{\mathbb{L}} S,$$

where  $\text{mod}_0^{\mathbb{L}} S$  denotes the Serre subcategory of  $\text{mod}^{\mathbb{L}} S$  of those modules of finite length (=finite  $k$ -dimension). We refer to the  $\mathbb{L}$ -graded algebra  $S$  as the *projective coordinate algebra* of  $\mathbb{X}$ .

The action of  $\mathbb{L}$  on  $\text{mod}^{\mathbb{L}} S$  induces an action on  $\text{coh } \mathbb{X}$ , given by *line bundle twists*,  $\sigma(\vec{x}): E \mapsto E(\vec{x})$  and thus determines a subgroup  $\text{Pic}(\mathbb{X})$ , called the *Picard group*, of the automorphism group of  $\text{coh } \mathbb{X}$ .

Each coherent sheaf has a decomposition  $X = X_+ \oplus X_0$  where  $X_0$  has finite length and  $X_+$  has no simple subobject, that is, is a *vector bundle*. By  $\text{vect } \mathbb{X}$  (resp.  $\text{coh}_0 \mathbb{X}$ ) we denote the category of all vector bundles (resp. finite length sheaves).

**2.2. Serre duality.** The category  $\text{coh } \mathbb{X}$ , which is a connected, abelian and noetherian category, has *Serre-duality* in the form

$$\text{D Ext}_{\mathbb{X}}^1(X, Y) = \text{Hom}_{\mathbb{X}}(Y, X(\vec{\omega}))$$

for all  $X, Y \in \text{coh } \mathbb{X}$ . As a consequence  $\text{coh } \mathbb{X}$  has almost-split sequences and the autoequivalence  $\tau$  of  $\text{coh } \mathbb{X}$ , given by the line bundle twist with  $\vec{\omega}$ , serves as the *Auslander-Reiten translation*. In particular  $\tau$  preserves the rank.

**2.3. Line bundles.** By sheafification the  $\mathbb{L}$ -graded  $S$ -modules  $S(\vec{x})$  yield the *twisted structure sheaves*  $\mathcal{O}(\vec{x})$ . Due to graded factoriality of  $S$ , each line bundle  $L$  in  $\text{coh } \mathbb{X}$  has the form  $L = \mathcal{O}(\vec{x})$  for some  $\vec{x} \in \mathbb{L}$ . Further, for all  $\vec{x}, \vec{y} \in \mathbb{L}$  we obtain that

$$\text{Hom}_{\mathbb{X}}(\mathcal{O}(\vec{x}), \mathcal{O}(\vec{y})) = S_{\vec{y}-\vec{x}}.$$

This implies, in particular, that

$$\text{Hom}_{\mathbb{X}}(\mathcal{O}(\vec{x}), \mathcal{O}(\vec{y})) \neq 0 \quad \Leftrightarrow \quad \vec{x} \leq \vec{y} \text{ in } \mathbb{L}.$$

Invoking Serre duality we obtain the next result.

**Lemma 2.1.** *Let  $L$  be a line bundle and  $\vec{x}, \vec{y}$  be elements of  $\mathbb{L}$ . Then we have  $\text{Ext}^1(L(\vec{x}), L(\vec{y})) = 0 = \text{Ext}^1(L(\vec{y}), L(\vec{x})) = 0$  if and only if  $-\vec{c} \leq \vec{y} - \vec{x} \leq \vec{c}$ .  $\square$*

**2.4. Euler form.** The *Euler form*  $\langle -, - \rangle$  is the bilinear form on the *Grothendieck group*  $K_0(\mathbb{X})$  given on classes of objects by

$$\langle [E], [F] \rangle = \dim_k \operatorname{Hom}(E, F) - \dim_k \operatorname{Ext}^1(E, F).$$

As an abelian group  $K_0(\mathbb{X})$  is free of rank  $n = 2 + \sum_{i=1}^t (p_i - 1)$ .

**2.5. Rank, degree, slope and stability.** Rank and degree define linear forms  $\operatorname{rk}, \deg: K_0(\mathbb{X}) \rightarrow \mathbb{Z}$  characterized by the properties:  $\operatorname{rk}(\mathcal{O}(\vec{x})) = 1$  and  $\deg(\mathcal{O}(\vec{x})) = \delta(\vec{x})$  for each  $\vec{x}$  in  $\mathbb{L}$ . The rank (resp. degree) is strictly positive on non-zero vector bundles (resp. non-zero sheaves of finite length). Then for each non-zero sheaf  $X$  the quotient  $\mu(X) = \deg(X)/\operatorname{rk}(X)$  is a well defined member of  $\mathbb{Q} \cup \{\infty\}$ , called the *slope* of  $X$ . With these notations, we have that

$$(4) \quad \mu(\tau E) = \mu(E) + \delta(\vec{\omega})$$

for each object  $E$ .

A non-zero vector bundle  $E$  is called *stable* (resp. *semistable*) if  $\mu E' < \mu E$  (resp.  $\mu E' \leq \mu E$ ) holds for each proper subobject  $E'$  of  $E$ .

**2.6. Exceptional objects and perpendicular categories.** An object  $E$  in  $\operatorname{coh} \mathbb{X}$  (or more generally in its bounded derived category  $D^b(\operatorname{coh} \mathbb{X})$ ) is called *exceptional* if  $\operatorname{End}(E) = k$  and  $E$  has no self-extensions which by heredity of  $\operatorname{coh} \mathbb{X}$  amounts to  $\operatorname{Ext}^1(E, E) = 0$ . Each exceptional sheaf of finite length is concentrated in an *exceptional point*, say  $\lambda_i$  of weight  $p_i$ , and then has length at most  $p_i - 1$ . Each exceptional sheaf  $E$  is uniquely determined by its class in  $K_0(\operatorname{coh} \mathbb{X})$ , see [Hüb96] or [Mel04].

An *exceptional sequence*  $E_1, \dots, E_n$ , say in  $D^b(\operatorname{coh} \mathbb{X})$ , consists of exceptional objects such that, whenever  $j > i$ , we have  $\operatorname{Hom}(E_j, E_i[m]) = 0$  for all integers  $m$ . If, moreover,  $n$  equals the rank of the Grothendieck group of  $\operatorname{coh} \mathbb{X}$ , we call the sequence *complete*. The indecomposable summands of a (multiplicity-free) tilting object  $T$  in  $\operatorname{coh} \mathbb{X}$  (or  $D^b(\operatorname{coh} \mathbb{X})$ ) can always be arranged as a complete exceptional sequence.

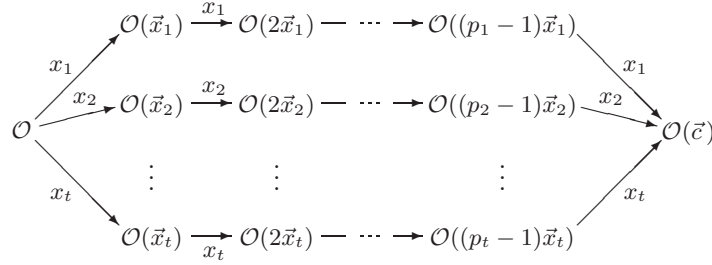
If  $E$  is exceptional in  $\operatorname{coh} \mathbb{X}$ , its right perpendicular category  $E^\perp$  consists of all objects  $X$  from  $\operatorname{coh} \mathbb{X}$  satisfying  $\operatorname{Hom}(E, X) = 0 = \operatorname{Ext}^1(E, X)$ . It is again an abelian hereditary category with Serre duality. If  $\mathcal{D} = D^b(\operatorname{coh} \mathbb{X})$  denotes the bounded *derived* category, it is also possible to form the right perpendicular category of  $E^{\perp_{\mathcal{D}}}$ , formed in  $\mathcal{D}$ , consisting of all objects  $X$  of  $\mathcal{D}$  satisfying  $\operatorname{Hom}(E, X[m]) = 0$  for each integer  $m$ . As is easily seen,  $E^{\perp_{\mathcal{D}}}$  equals  $D^b(E^\perp)$ .

**2.7. Coxeter polynomials.** Any triangulated category  $\mathcal{T}$  with a tilting object satisfies Serre duality in the form  $D \operatorname{Hom}(X, Y[1]) = \operatorname{Hom}(Y, \tau X)$  for a self-equivalence  $\tau$  of  $\mathcal{T}$ . On the Grothendieck group  $K_0(\mathcal{T})$ , the equivalence  $\tau$  induces an invertible  $\mathbb{Z}$ -linear map, the *Coxeter transformation* of  $\mathcal{T}$ . Its characteristic polynomial is called the *Coxeter polynomial* of  $\mathcal{T}$ . Typical instances for  $\mathcal{T}$  are the (bounded) derived category  $D^b \mathcal{H}$ , where  $\mathcal{H}$  is a hereditary category with Serre duality, or the (bounded) derived category  $D^b(\operatorname{mod} A)$  of modules over a finite dimensional algebra of finite global dimension.



**2.8. Tilting objects.** An object  $T \in \text{coh } \mathbb{X}$  is called a *tilting sheaf* if  $\text{Ext}^1(T, T) = 0$  and  $T$  generates the category  $\text{coh } \mathbb{X}$  homologically, in the sense that  $\text{Hom}(T, X) = 0 = \text{Ext}^1(T, X)$  implies  $X = 0$ . If further  $T$  is a vector bundle, it is called a *tilting bundle*. In the terminology of [LM96] the endomorphism algebras of tilting bundles are *concealed canonical* and thus by [LM96] have a sincere separating tubular family (subcategory) of stable tubes. Moreover, by [LdlP99] the existence of such a separating subcategory characterizes concealed canonical algebras.

The line bundles  $\mathcal{O}(\vec{x})$ ,  $0 \leq \vec{x} \leq \vec{c}$ , yield the *canonical tilting bundle*  $T_{\text{can}} = \bigoplus_{0 \leq \vec{x} \leq \vec{c}} \mathcal{O}(\vec{x})$  for  $\text{coh } \mathbb{X}$ . Its endomorphism ring is given by the quiver



with the *canonical relations*

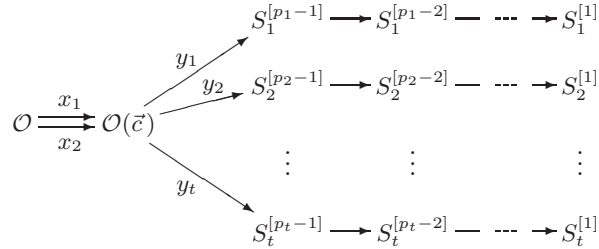
$$(5) \quad x_i^{p_i} = x_2^{p_2} - \lambda_i x_1^{p_1}, \quad i = 3, \dots, t.$$

This algebra is the *canonical algebra associated with*  $\mathbb{X}$ .

Another tilting sheaf in  $\text{coh } \mathbb{X}$ , competing with  $T_{\text{can}}$  for the role to be ‘the most natural tilting sheaf’ is the *squid tilting sheaf*  $T_{\text{squid}}$ , which we are going to define now. For each  $i$  from 1 to  $t = t(\mathbb{X})$  there is exactly one simple sheaf  $S_i$ , concentrated in  $\lambda_i$  satisfying  $\text{Hom}(\mathcal{O}, S_i) \neq 0$ . Note for this purpose that  $\lambda_1 = \infty$  and  $\lambda_2 = 0$ . Moreover, there exists a sequence of exceptional objects of finite length together with epimorphisms

$$B_i : S_i^{[p_i-1]} \twoheadrightarrow S_i^{[p_i-2]} \twoheadrightarrow \dots \twoheadrightarrow S_i^{[1]} = S_i,$$

where  $S_i^{[j]}$  has length  $j$  and top  $S_i$ . The direct sum of  $\mathcal{O}$ ,  $\mathcal{O}(\vec{c})$  and all the  $S_i^{[j]}$ ,  $i = 1, \dots, t$ ,  $j = 1, \dots, p_i - 1$ , then forms the tilting sheaf  $T_{\text{squid}}$



whose endomorphism algebra is the *squid algebra*  $C_{\text{squid}}$  associated with  $\mathbb{X}$ . It is given by the above quiver and subject to the relations

$$y_1 x_1 = 0, \quad y_2 x_2 = 0, \quad y_i(x_2 - \lambda_i x_1) = 0 \text{ for } i = 3, \dots, t.$$

A less known tilting object, actually a tilting complex  $T_{\text{cox}}$  in  $D^b(\text{coh } \mathbb{X})$ , is displayed below. It is called *Coxeter-Dynkin configuration of canonical type* and exists for



$t(\mathbb{X}) \geq 2$ . As the squid it consists of two line bundles and of  $t = t(\mathbb{X})$  branches of finite length sheaves, up to translation in  $D^b(\text{coh } \mathbb{X})$ . Following [LdlP11], where the dual algebra is considered, its endomorphism ring  $C_{\text{cox}}$  is called a *Coxeter-Dynkin algebra of canonical type*. Such algebras, actually their underlying bigraphs, play a prominent role in singularity theory, compare for instance [Ebe07].

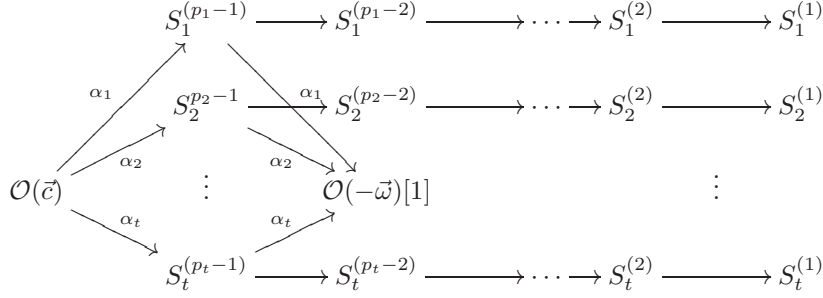


FIGURE 2.1. Coxeter-Dynkin algebra of canonical type

The endomorphism algebra  $C_{\text{cox}}$  of  $T_{\text{cox}}$  is given by the above ‘quiver’ with the two relations

$$(6) \quad \sum_{i=2}^t \alpha_i^2 = 0 \quad \text{and} \quad \alpha_1^2 = \sum_{i=3}^t \lambda_i \alpha_i^2.$$

It is remarkable that  $C_{\text{cox}}$  is Schurian for  $t(\mathbb{X}) = 3$ . Moreover, for  $t(\mathbb{X}) \geq 5$  the number of relations is strictly less than for the canonical algebra or the squid algebra. In the tubular case, and only in there,  $C_{\text{cox}}$  can be realized as the endomorphism ring of a tilting sheaf, actually as the endomorphism ring of a tilting bundle, see Section 3.6 for an interesting extremal property of these algebras.

**2.9. Tubular mutations.** Assume  $\mathbb{X}$  has Euler characteristic zero. Tubular mutations are distinguished self-equivalences of  $D^b(\text{coh } \mathbb{X})$  playing a key role in the classification of indecomposable objects. By tilting they are related to Ringel’s shrinking functors from [Rin84]. Their formal definition is due to [LM93]. From different perspectives, the subject is also treated in [Mel97], [LdlP99], [Kus09], [Len07]. For quick information we recommend the survey in [Mel04].

The tubular mutation  $\rho : D^b \text{coh } \mathbb{X} \rightarrow D^b \text{coh } \mathbb{X}$  is a triangle equivalence that is given on indecomposable objects of slope  $\mu X > 0$  by the universal extension

$$(7) \quad 0 \longrightarrow \bigoplus_{j=1}^{\bar{p}} \text{Ext}^1(X, \tau^j \mathcal{O}) \otimes_k \tau^j \mathcal{O} \longrightarrow \rho X \longrightarrow X \longrightarrow 0,$$

see [LM93] or [Len07, Section 10.3]. Another self-equivalence of  $D^b \text{coh } \mathbb{X}$ , actually also a tubular mutation, is given by the line bundle shift  $\sigma(X) = X(\vec{x}_t)$ , where  $\vec{x}_t$  belongs to the largest weight of  $\mathbb{X}$ , thus  $\delta(\vec{x}_t) = 1$ . On the pair  $(d, r) = (\deg X, \text{rk } X)$  the actions induced by  $\sigma$  and  $\rho$  are given by the right multiplication with the matrix  $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$  resp.  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ . In particular,  $\sigma$  (resp.  $\rho$ ) preserves the rank (resp. the

degree). Further,  $\sigma$  (resp.  $\rho$ ) induce actions on slopes, given by the fractional linear transformations  $q \mapsto 1 + q$  (resp.  $q \mapsto q/(1 + q)$ ).

The self-equivalences  $\sigma$  and  $\lambda = \rho^{-1}$  are conjugate, actually

$$(8) \quad \lambda = (\lambda\sigma)^{-1}\sigma(\lambda\sigma)$$

which follows from the braid relation  $\sigma\lambda\sigma = \lambda\sigma\lambda$ , see for instance [LM00, prop. 6.2].

**2.10. Hübner reflections.** A useful tool to construct new tilting sheaves from given ones is by means of mutations (more precisely, by *Hübner reflections*). We recall the relevant facts from [Hüb96], see also [Hüb97]. Let  $T$  be a (multiplicity-free) tilting sheaf on a weighted projective line  $\mathbb{X}$  and assume that  $T = T' \oplus E$  with  $E$  indecomposable. Then there exists exactly one indecomposable object  $E^*$  not isomorphic to  $E$  such that  $T^* = T' \oplus E^*$  is again a tilting sheaf. We say that  $T^*$  is the mutation of  $T$  at the vertex (corresponding to)  $E$ . In more detail, let  $Q$  be the quiver of the endomorphism algebra  $A$  of  $T$ . Let  $T_1, \dots, T_n$  be the (nonisomorphic) indecomposable summands of  $T$  and let  $U_1, \dots, U_n$  be the corresponding simple  $A$ -modules, viewed as members of  $D^b(\text{coh } \mathbb{X})$  where we identify  $T_i$  with the  $i$ -th indecomposable projective  $A$ -module. A vertex is called a *formal source* (resp. a *formal sink*) if  $U_i$  belongs to  $\text{coh } \mathbb{X}$  (to  $(\text{coh } \mathbb{X})[1]$ , respectively). Each source (resp. sink) of  $Q$  is a formal source (resp. a formal sink). Moreover, each vertex  $i$  of  $Q$  is either a formal source or a formal sink. Assume that  $i$  is a formal sink. Then there exists an exact sequence

$$(9) \quad 0 \longrightarrow T_i^* \xrightarrow{u} \bigoplus_{j=1}^n T_j^{\kappa_j} \longrightarrow T_i \longrightarrow 0$$

where  $\kappa_j$  denote the number of arrows from  $j$  to  $i$  in  $Q$ , and where  $u$  collects these arrows. Moreover, we have  $\text{Ext}^1(T_i, T_i^*) = k$  and  $\text{Ext}^1(T_i^*, T_i) = 0$ . The case of a formal source is dual.

### 3. PROOFS AND MORE

We fix a tilting bundle  $T$  on  $\mathbb{X}$  and denote by  $A = \text{End}(T)$  its endomorphism ring. We recall the previous convention to consider  $\text{mod } A$  as a full subcategory of  $\text{coh } \mathbb{X} \vee \text{coh } \mathbb{X}[1]$ , and use the notation  $S_1, \dots, S_n$  for the corresponding simple  $A$ -modules. Let  $\mathbf{w}$  denote the class of any homogeneous simple sheaf  $S_0$ . Note that  $\text{rk}(x) = \langle x, \mathbf{w} \rangle = -\langle \mathbf{w}, x \rangle$  for each  $x \in K_0(\mathbb{X})$ .

**Lemma 3.1** ([Hüb96, Prop. 4.26]). *Assume  $T$  is a tilting bundle on  $\mathbb{X}$ . With the preceding notations we have*

$$\sum_{i=1}^n \text{rk}(T_i) [S_i] = \mathbf{w}, \quad \sum_{i=1}^n \text{rk}(S_i) [T_i] = -\mathbf{w}.$$

*Proof.* Indeed, the Euler form  $\langle -, - \rangle$  satisfies  $\langle [T_i], [S_j] \rangle = \delta_{ij}$ . Expressing  $\mathbf{w}$  in the basis of the simples  $\mathbf{w} = \sum_{i=1}^n \alpha_i [S_i]$ , we get

$$\text{rk}(T_j) = \langle [T_j], \mathbf{w} \rangle = \sum_{i=1}^n \alpha_i \langle [T_j], [S_i] \rangle = \sum_{i=1}^n \alpha_i \delta_{ij} = \alpha_j,$$

and the first formula follows. Expressing  $-\mathbf{w}$  in the base of the projectives  $-\mathbf{w} = \sum_{i=1}^n \beta_i [T_i]$ , we get

$$\mathrm{rk}(S_j) = \langle [S_j], \mathbf{w} \rangle = -\langle \mathbf{w}, [S_j] \rangle = \sum_{i=1}^n \beta_i \langle [T_i], [S_j] \rangle = \beta_j.$$

This shows the second formula.  $\square$

**3.1. Maximal number of central simples.** We are now going to prove Theorem 1.4. Denote by  $S_1, \dots, S_n$  the simple  $\mathrm{End}(T)$ -modules corresponding to the projectives  $T_1, \dots, T_n$  respectively. We may assume that  $T_1, \dots, T_n$  form an exceptional sequence, a fact implying that the simples  $S_n, \dots, S_1$  form an exceptional sequence in the reverse direction.

To prove the first claim of the theorem, we observe that the vertex associated to  $T_1$ , resp. to  $T_n$  is a source, resp. a sink, of the quiver of  $A = \mathrm{End}(T)$ . Hence  $S_1 = T_1$  is simple projective over  $A$  of positive rank and  $S_n = \tau T_n[1]$  is simple injective over  $A$  of negative rank. Since central simple  $A$ -modules have rank zero, we conclude that the number of central simple modules is at most  $n-2$ . The bound  $n-2$  is actually attained for the canonical tilting bundle  $T_{\mathrm{can}}$ : Then  $0 \rightarrow \mathcal{O}((j-1)\vec{x}_i) \rightarrow \mathcal{O}(j\vec{x}_i) \rightarrow S_{i,j} \rightarrow 0$  is a projective resolution of the simple  $A$ -module  $S_{i,j}$  associated to the projective  $\mathcal{O}(j\vec{x}_i)$  for  $j = 1, \dots, p_i - 1$ . Hence  $\mathrm{rk} S_{i,j} = \mathrm{rk} \mathcal{O}(j\vec{x}_i) - \mathrm{rk} \mathcal{O}((j-1)\vec{x}_i) = 1 - 1 = 0$  showing that  $S_{i,j}$  is a central simple  $A$ -module.

We next assume that  $T$  is a tilting bundle with  $n-2$  central simple modules over  $A$ . As before we conclude that  $T_1$  (resp.  $T_n$ ) is a simple projective (resp. simple injective)  $A$ -module of positive (resp. negative) rank. Hence the simple  $A$ -modules  $S_2, \dots, S_{n-1}$  have rank zero.

Applying Lemma 3.1 we obtain  $\mathrm{rk}(S_1)[T_1] + \mathrm{rk}(S_n)[T_n] = -\mathbf{w}$ . Since  $S_1 = T_1$  and  $S_n = \tau T_n[1]$  we have  $\mathrm{rk}(S_1) = \mathrm{rk}(T_1)$  and  $\mathrm{rk}(S_n) = -\mathrm{rk}(T_n)$ . Consequently

$$(10) \quad \mathrm{rk}(T_1)[T_1] + \mathbf{w} = \mathrm{rk}(T_n)[T_n]$$

Applying the rank function to (10) we get  $(\mathrm{rk} T_1)^2 = (\mathrm{rk} T_n)^2$ , and conclude that  $\mathrm{rk} T_1 = \mathrm{rk} T_n$  since both values are positive. Calling this common value  $\rho$ , then (10) implies that  $\mathbf{w} = \rho([T_n] - [T_1])$ . Since  $\mathbf{w}$  is indivisible in the Grothendieck group  $K_0(\mathrm{coh} \mathbb{X})$  we further get  $\rho = 1$ . Hence  $T_1 = L$  and  $T_n$  are line bundles and now (10) implies that  $T_n = L(\vec{c})$ .

If  $\deg(S_1) = q$ , it follows from (4) that  $\deg(S_n) = \deg(\tau T_n[1]) = -\deg(\tau T_n) = -(\deg(L(\vec{c})) + \delta(\vec{\omega})) = -(q + \bar{p} + \delta(\vec{\omega}))$ . Invoking  $\deg \mathbf{w} = \bar{p}$  and additionally Lemma 3.1 we obtain

$$\begin{aligned} \bar{p} &= \mathrm{rk} T_1 \deg S_1 + \mathrm{rk} T_n \deg S_n + \sum_{h=2}^{n-1} \mathrm{rk} T_h \deg S_h \\ &= -(\bar{p} + \delta(\vec{\omega})) + \sum_{h=2}^{n-1} \mathrm{rk} T_h \deg S_h. \end{aligned}$$

Hence we get

$$(11) \quad 2\bar{p} + \delta(\vec{\omega}) = \sum_{h=2}^{n-1} \mathrm{rk} T_h \deg S_h \geq \sum_{h=2}^{n-1} \deg S_h.$$

By assumption the  $A$ -modules  $S_i$ ,  $i = 2, \dots, n-1$ , are simple exceptional sheaves of rank zero. Since further  $S_{n-1}, \dots, S_2$  form an exceptional sequence, each exceptional tube of  $\text{coh}_0 \mathbb{X}$  with  $p_j$  simple sheaves can contain at most  $p_j - 1$  simple  $A$ -modules. Since  $\sum_{j=1}^t (p_j - 1) = n - 2$ , our assumption on the number of central simples implies that each exceptional tube of rank  $p_j$  contains exactly  $p_j - 1$  of them. Using further that  $\deg(S_h) \geq \frac{p}{p_j}$  if  $S_h$ ,  $h = 2, \dots, n-1$ , belongs to an exceptional tube of rank  $p_j$ , we thus obtain:

$$(12) \quad \sum_{h=2}^{n-1} \deg S_h \geq \sum_{j=1}^t (p_j - 1) \frac{\bar{p}}{p_j} = t \cdot \bar{p} + \sum_{j=1}^t \frac{1}{p_j} = 2\bar{p} + \delta(\vec{\omega}).$$

This implies that inequality (11) is indeed an equality, which proves that  $\text{rk } T_i = 1$  for all  $i = 1, \dots, n$ .

Thus  $T$  is a direct sum of line bundles  $T_i = L(\vec{y}_i)$  and, moreover,  $T_1 = L$  and  $T_n = L(\vec{c})$ . Applying Lemma 2.1 to the pairs  $L$ ,  $L(\vec{y}_i)$  and  $L(\vec{y}_i)$ ,  $L(\vec{c})$ , we then obtain  $0 \leq \vec{y}_i \leq \vec{c}$ . This shows that  $T = \bigoplus_{0 \leq \vec{x} \leq \vec{c}} L(\vec{x})$  is the canonical tilting bundle up to a line bundle twist, and thus finishes the proof of Theorem 1.4.

**3.2. Maximal number of line bundles.** We now prove Theorem 1.1. Since  $T = \bigoplus_{i=1}^n T_i$  is tilting, we obtain  $[\mathcal{O}] = \sum_{i=1}^n m_i [T_i]$  with  $m_i \in \mathbb{Z}$ . Passing to ranks we obtain that the common rank  $r$  of the  $T_i$  divides 1. Hence  $r = 1$  follows, and thus  $T = \bigoplus_{\vec{y} \in J} \mathcal{O}(\vec{y})$  for some subset  $J \subset \mathbb{L}$  of cardinality  $n = 2 + \sum_{i=1}^t (p_i - 1)$  where  $\underline{p} = (p_1, \dots, p_t)$  is the weight sequence of  $\mathbb{X}$ . By means of a line bundle twist, we may assume that (i)  $0 \in J$  and, moreover, (ii)  $0 \leq \delta(\vec{x})$  for all  $\vec{x} \in J$ .

Lemma 2.1 implies (iii)  $-\vec{c} \leq \vec{x} \leq \vec{c}$  for each  $\vec{x} \in J$ . Invoking the normal form  $\vec{x} = \sum_{i=1}^t \ell_i \vec{x}_i + \ell \vec{c}$  with  $0 \leq \ell_i < p_i$  and  $\ell \in \mathbb{Z}$ , conditions (ii) and (iii) imply that  $\ell \in \{-1, 0, 1\}$ . Note that  $\ell \in \{0, 1\}$  implies that  $\vec{x} = a\vec{x}_i$  for some  $i = 1, \dots, t$  and  $0 \leq a \leq p_i$ . If  $\ell = -1$ , then the inequality  $0 \leq \vec{x} + \vec{c} = \sum_{i=1}^t \ell_i \vec{x}_i \leq 2\vec{c}$  shows that exactly two of the summands  $\ell_i \vec{x}_i$  are non-zero. Hence  $\vec{x} + \vec{c} = \ell_i \vec{x}_i + \ell_j \vec{x}_j$  with  $i \neq j$  and then  $\vec{x} = a_i \vec{x}_i - b_j \vec{x}_j$  with  $0 < a_i < p_i$  and  $0 < b_j < p_j$ . In the first case, where  $\ell \in \{0, 1\}$ , we call  $\vec{x}$  *unmixed*, in the second case, where  $\ell = -1$ ,  $\vec{x}$  is called *mixed*.

We distinguish the two cases (a)  $\vec{c} \in J$  and (b)  $\vec{c} \notin J$ .

*Case (a):* If  $\vec{c}$  belongs to  $J$ , then Lemma 2.1 implies  $0 \leq \vec{x} \leq \vec{c}$  for each  $\vec{x} \in J$  implying that  $T = T_{\text{can}}$  by cardinality reasons.

*Case (b):* If  $\vec{c}$  does not belong to  $J$ , then  $J$  contains a mixed element, say,  $\vec{y} = a_1 \vec{x}_1 - a_2 \vec{x}_2$  with  $0 < a_1 < p_1$  and  $0 < a_2 < p_2$ .

We are going to show that then  $t(\mathbb{X}) = 2$  and first claim that  $J \subset \mathbb{Z}\vec{x}_1 + \mathbb{Z}\vec{x}_2$ . Indeed let  $0 \neq \vec{x} \in J$ , say,  $\vec{x} = b_i \vec{x}_i - b_j \vec{x}_j$  where  $i \neq j$ ,  $i, j = 1, \dots, t$ ,  $0 < b_i < p_i$ , and  $0 \leq b_j < p_j$ . Note that  $b_i = 0$  is impossible since  $\vec{x} \neq 0$  and  $\delta(\vec{x}) \geq 0$ . By Lemma 2.1 we get  $0 \leq \vec{y} - \vec{x} + \vec{c} \leq 2\vec{c}$ , hence

$$0 \leq a_1 \vec{x}_1 + b_j \vec{x}_j + (p_2 - a_2) \vec{x}_2 - b_i \vec{x}_i \leq 2\vec{c}.$$

Since  $i \neq j$  this is only possible if  $i \in \{1, 2\}$ . If  $b_j = 0$ , then  $\vec{x} = b_i \vec{x}_i$  belongs to  $\mathbb{Z}\vec{x}_1 + \mathbb{Z}\vec{x}_2$ . If  $b_j \neq 0$ , then reversing the roles of  $\vec{x}$  and  $\vec{y}$  in the preceding argument we get that also  $j$  belongs to  $\{1, 2\}$ . Summarizing we obtain that  $J \subset \mathbb{Z}\vec{x}_1 + \mathbb{Z}\vec{x}_2$ .

Finally assume that  $t(\mathbb{X}) \geq 3$ . Let  $S_3$  be a simple sheaf concentrated in the third exceptional point of weight  $p_3$  and such that  $\text{Hom}(\mathcal{O}, S_3) = 0$ . Since  $S_3(\vec{x}_i) = S_3$  for each  $i \neq 3$  we obtain that  $\text{Hom}(\mathcal{O}(\vec{x}), S_3) = 0$  for each  $\vec{x}$  from  $\mathbb{Z}\vec{x}_1 + \mathbb{Z}\vec{x}_2$ . In particular, we get  $\text{Hom}(T, S_3) = 0$ . Because  $T$  is a vector bundle, we further obtain  $\text{Ext}^1(T, S_3) = 0$ , contradicting the assumption that  $T$  is tilting. This finishes the proof of Theorem 1.1.

**3.3. Homogeneity.** Let  $A$  be a finite dimensional  $k$ -algebra, and  $E$  a finite dimensional right  $A$ -module. Then the  $k$ -algebra

$$A[E] = \begin{pmatrix} A & 0 \\ E & k \end{pmatrix}$$

is called the *one-point extension* of  $A$  by  $E$ . The proof of Theorem 1.2 is based on the next proposition.

**Proposition 3.2.** *Let  $T$  be a tilting sheaf in  $\text{coh } \mathbb{X}$  with indecomposable summands  $T_1, \dots, T_n$ . Consider the following properties:*

- (a)  *$T$  is homogeneous in the abelian category  $\text{coh } \mathbb{X}$ .*
- (b)  *$T$  is homogeneous in the triangulated category  $\text{D}^b(\text{coh } \mathbb{X})$ .*
- (c) *The perpendicular categories  $T_i^\perp$ , formed in  $\text{coh } \mathbb{X}$ , are pairwise derived equivalent.*
- (d) *The Coxeter polynomial  $\psi'_i$  of  $T_i^\perp$  does not depend on  $i = 1, \dots, n$ .*
- (e) *The Coxeter polynomial  $\bar{\psi}_i$  of the one-point extension  $A[P_i]$  of  $A = \text{End}(T_i)$  by the indecomposable projective  $A$ -module  $P_i$ , corresponding to  $T_i$ , does not depend on  $i = 1, \dots, n$ .*

*Then we always have the implications  $(a) \implies (b) \implies (c) \implies (d) \iff (e)$ . Moreover, if  $\mathbb{X}$  is not tubular, the indecomposable summands of  $T$  are line bundles, forcing the equivalence of (a) to (e).*

*Proof.* (a)  $\implies$  (b): Each self-equivalence of  $\text{coh } \mathbb{X}$  extends to a self-equivalence of  $\text{D}^b(\text{coh } \mathbb{X})$ .

(b)  $\implies$  (c): With  $\mathcal{D} = \text{D}^b(\text{coh } \mathbb{X})$ , the right perpendicular category  $T_i^{\perp \mathcal{D}}$  formed in  $\mathcal{D}$ , equals the derived category of  $T_i^\perp$ .

(c)  $\implies$  (d): The Coxeter polynomial is preserved under derived equivalence.

(d)  $\iff$  (e): By [Len99, Prop. 18.3 and Cor. 18.2], see also [LdlP08, Prop. 4.5], the Coxeter polynomials  $\psi'_i$  of  $T_i^\perp$  and  $\bar{\psi}_i$  of the one-point extension  $A[P_i]$  are related by the reciprocity formula

$$(13) \quad P_{T_i} = \frac{\psi - x\psi'_i}{\psi} = \frac{\bar{\psi}_i - x\psi}{\psi},$$

where  $\psi$  denotes the Coxeter polynomial of  $\text{coh } \mathbb{X}$ , and

$$(14) \quad P_{T_i} = \sum_{n=0}^{\infty} \langle [T_i], [\tau^n T_i] \rangle x^n$$

denotes the *Hilbert-Poincaré series* of  $T_i$ . The equivalence of conditions (d) and (e) is now implied by formula (13), thus finishing the proof of the first claim.

We next assume that  $\mathbb{X}$  is not tubular, and that (d) or (e) holds which forces by (13) all the  $P_{T_i}$  to be equal. We claim that all the  $T_i$  have the same rank. Let  $\alpha_m$  denote the  $m$ -th coefficient of  $P_{T_i}$ . Denoting, as usual, by  $\bar{p}$  the least common multiple of the weights, we have for each  $x$  in  $K_0(\text{coh } \mathbb{X})$

$$\tau^{\bar{p}}x = x + \text{rk}(x) \delta(\bar{\omega}) \mathbf{w},$$

where  $\mathbf{w}$  denotes the class of any ordinary simple sheaf on  $\mathbb{X}$ . This is implied by the formula  $\bar{p}\bar{\omega} = \delta(\bar{\omega})\bar{c}$  from [GL87]. It follows that

$$\alpha_{\bar{p}} - \alpha_0 = \text{rk}(T_i) \delta(\bar{\omega}) \langle [T_i], \mathbf{w} \rangle = \text{rk}(T_i)^2 \delta(\bar{\omega}),$$

and thus  $\text{rk}(T_i)^2 \delta(\bar{\omega})$  does not depend on  $i$ . By our assumption  $\delta(\bar{\omega}) \neq 0$ , then the rank of  $T_i$  does not depend on  $i = 1, \dots, n$ . Therefore by Theorem 1.1 all the  $T$  are line bundles, forcing  $T$  to be homogeneous.  $\square$

*Proof of Theorem 1.2.* By [LM00] each self-equivalence of  $\text{coh } \mathbb{X}$  is rank preserving. Hence the indecomposable summands of each homogeneous tilting sheaf have the same rank. By Theorem 1.1 thus property (i) implies the claim. For the remaining properties (ii) to (iv) the claim follows from Proposition 3.2.  $\square$

**3.4. Maximal amount of bijections.** We recall that a module  $G$  over a finite dimensional  $k$ -algebra  $A$  is called *generic* if (i)  $G$  is indecomposable, (ii)  $G$  has finite length over  $\text{End}(G)$ , and (iii)  $G$  has infinite  $k$ -dimension. Now let  $A$  be an *almost concealed canonical* algebra, that is, the endomorphism algebra of a tilting sheaf  $T$  on a weighted projective line  $\mathbb{X}$ . The injective hull  $\mathbb{E}(\mathcal{O})$  of the structure sheaf in the category  $\text{Qcoh } \mathbb{X}$  of quasicoherent sheaves on  $\mathbb{X}$  equals the *sheaf of rational functions*  $\mathcal{K}$ . Under the equivalence  $\text{RHom}(T, -) : \text{D}^b(\text{Qcoh } \mathbb{X}) \rightarrow \text{D}^b(\text{Mod } A)$  the sheaf  $\mathcal{K}$  corresponds to a generic  $A$ -module  $G$ , called the  *$T$ -distinguished generic  $A$ -module*. It is known, compare [Len97a, RR06], that  $G$  is the unique generic  $A$ -module if  $\chi_{\mathbb{X}} \neq 0$ . In the tubular case  $\chi_{\mathbb{X}} = 0$  the situation is more complicated, since there exists a rational family  $(G^{(q)})_q$  of generic  $A$ -modules, indexed by a set of rational numbers; for details we refer to [Len97a]. In the proper formulation our next result extends also to the generic modules  $G_q$ . Details are left to the reader. Here, we restrict to the  $T$ -distinguished case.

We now prove Theorem 1.3. We view  $G = \mathcal{K}$  as a (contravariant) representation of the quiver of  $A$ . First we show that each arrow  $\alpha : u \rightarrow v$  induces a monomorphism or an epimorphism  $G_\alpha : G_v \rightarrow G_u$ . It follows from [Len97a] that the endomorphism ring of  $G$  equals the rational function field  $K = k(x)$  and, moreover,  $\text{rk } T_u = \dim_K(G_u)$  for each vertex  $u$  of the quiver of  $A$ . By a result of Happel-Ringel [HR82, Lemma 4.1], the map  $T_\alpha : T_u \rightarrow T_v$  is a monomorphism or an epimorphism since  $\text{Ext}^1(T_v, T_u) = 0$ . Since  $G$  is injective in the category of quasi-coherent sheaves on  $\mathbb{X}$  this yields that  $G_\alpha = \text{Hom}(\alpha, G) : G_v \rightarrow G_u$  is an epimorphism or a monomorphism of  $K$ -vector spaces. Hence  $G_\alpha$  is bijective if and only if  $\dim_K G_v = \dim_K G_u$ , that is, if and only if  $\text{rk } T_u = \text{rk } T_v$ . In particular, for the canonical tilting bundle all  $G_\alpha$  are bijective. Conversely, assuming that all  $G_\alpha$  are bijective, connectedness of the quiver of  $A$  implies that all  $T_u$ ,  $u = 1, \dots, n$ , have the same rank, and then Theorem 1.1 implies that  $T$  equals  $T_{\text{can}}$  up to a line bundle twist.

**3.5. Maximal width.** The proof of Theorem 1.5 is based on the following proposition.

**Proposition 3.3.** *Let  $E$  and  $F$  be non-zero vector bundles on a weighted projective line  $\mathbb{X}$  of arbitrary weight type. Then the following properties hold:*

- (i) *If  $\mu F - \mu E > \delta(\vec{c} + \vec{\omega})$  then  $\text{Hom}(E, F)$  is non-zero.*
- (ii) *If  $\text{Ext}^1(F, E) = 0$  then  $\mu F - \mu E \leq \delta(\vec{c})$ .*

*Proof.* Property (i) is shown in [LdlP97, Thm. 2.7]. Further, property (ii) follows from (i) by Serre duality.  $\square$

It amounts to a significant restriction for  $E$  and  $F$  to attain the bound for the slope in part (ii).

**Corollary 3.4.** *Let  $E$  and  $F$  be non-zero vector bundles satisfying  $\text{Ext}^1(F, E) = 0$  with slope difference  $\mu F - \mu E = \bar{p}$ . Then  $E$  and  $F$  are semistable.*

In particular, if  $E$  and  $F$  are indecomposable and  $\chi_{\mathbb{X}} < 0$  then  $E$  and  $F$  are quasi-simple in their respective Auslander-Reiten components which have type  $\mathbb{Z}\mathbb{A}_{\infty}$ .

*Proof.* By symmetry it suffices to show that each non-zero subobject  $F'$  of  $F$  has a slope  $\mu F' \leq \mu F$ . Indeed, since the category  $\text{coh } \mathbb{X}$  is hereditary, vanishing of  $\text{Ext}^1(F, E)$  implies that  $\text{Ext}^1(F', E) = 0$ . By Proposition 3.3 thus  $\mu F' - \mu E \leq \bar{p}$ . This forces  $\mu F' \leq \mu F$  and proves the semistability of  $F$ .  $\square$

*Proof of Theorem 1.5.* Let  $T$  be a tilting bundle on  $\mathbb{X}$  with  $\mu T_1 \leq \dots \leq \mu T_n$ . Since  $0 = \text{Ext}^1(T_n, T_1) = \text{D Hom}(T_1, T_n(\vec{\omega}))$  we get by the preceding proposition that  $\mu T_n(\vec{\omega}) - \mu T_1 \leq \bar{p} + \delta(\vec{\omega})$ . Since  $\mu T_n(\vec{\omega}) = \mu T_n + \delta(\vec{\omega})$  we conclude that  $\mu T_n - \mu T_1 \leq \bar{p}$ , showing that the width  $w(T)$  is bounded by  $\bar{p}$ .

The bound  $\bar{p}$  for the width is clearly attained for the canonical tilting bundle since  $0 = \mu \mathcal{O}$  and  $\mu \mathcal{O}(\vec{c}) = \bar{p}$ .

We now assume that  $\chi_{\mathbb{X}} \geq 0$  and that  $T$  is a tilting bundle on  $\mathbb{X}$  with width  $w(T) = \bar{p}$ . We set  $F = T_n(-\vec{c})$  and observe that  $\mu F = \mu T_1$  and  $[T_n] = [F] + \text{rk}(F)\mathbf{w}$  in  $K_0(\text{coh } \mathbb{X})$ . Since  $\text{Hom}(T_n, T_1) = 0$  by semistability and  $\text{Ext}^1(T_n, T_1) = 0$  we get  $0 = \langle [T_n], [T_1] \rangle = -\langle [T_1], [T_n(\vec{\omega})] \rangle$  and therefore

$$\begin{aligned} 0 &= \langle [T_1], [T_n(\vec{\omega})] \rangle = \langle [T_1], [F(\vec{\omega})] \rangle + \text{rk } F \langle [T_1], \mathbf{w} \rangle \\ &= \langle [T_1], [F(\vec{\omega})] \rangle + \text{rk } T_1 \text{rk } F \\ (15) \quad &= -\langle [F], [T_1] \rangle + \text{rk } T_1 \text{rk } T_n. \end{aligned}$$

Since  $T_1$  and  $T_n$  have positive rank this implies that  $\text{Hom}(F, T_1) \neq 0$ . We will show that  $F = T_1$  is a line bundle. If  $\chi_{\mathbb{X}} = 0$ , then  $F$  and  $T_1$  must lie in the same tube. By exceptionality, they further have a quasi-length less than the rank of the tube such that, in particular,  $\dim \text{Hom}(F, T_1) \leq 1$ . Then (15) implies that  $\text{Ext}^1(F, T_1) = 0$  and  $\dim \text{Hom}(F, T_1) = 1$ , thus  $\text{rk } T_1 = \text{rk } F = 1$  and finally  $F = T_1$ . If  $\chi_{\mathbb{X}} > 0$ , then  $F = T_1$  follows by stability since  $F$  and  $T_1$  have the same slope and  $\text{rk } T_1 = 1$  follows again by (15).

The argument also shows that  $T_n$  is the unique indecomposable summand of  $T$  of maximal slope  $\mu(T_n)$ , and dually  $T_1$  is the unique indecomposable summand of  $T$  having minimal slope. By semistability this implies  $\text{Hom}(T_i, T_1) = 0 = \text{Hom}(T_n, T_i)$  for all  $1 < i < n$  showing that  $T_1$  is a source and  $T_n$  a sink of the quiver of  $\text{End}(T)$ . We finally get that  $T$  is the canonical tilting bundle, up to a line bundle twist, by



applying Proposition 1.6 whose proof is given below. This will conclude the proof of Theorem 1.5.  $\square$

*Proof of Proposition 1.6.* Assertion (i) is a special case of Corollary 3.4.

Concerning assertion (ii) let  $L$  and  $L'$  be line bundle summands of  $T$ , corresponding to a sink (resp. a source) of the quiver of  $A$  and satisfying the maximality property  $\mu(L') - \mu(L) = \bar{p}$ . Since  $L'$  and  $L(\vec{c})$  have the same degree, we notice first that  $L' = L(\vec{c} + \vec{x})$  for some  $\vec{x}$  of degree zero. Because  $0 = \text{Ext}^1(L', L) = \text{D Hom}(L, L'(\vec{\omega}))$  we obtain  $\vec{c} + \vec{\omega} + \vec{x} \leq \vec{c} + \vec{\omega}$ , hence  $\vec{x} \leq 0$ . Since  $0 \geq \vec{x}$  and  $\vec{x}$  has degree zero, we obtain  $\vec{x} = 0$ , implying that  $L' = L(\vec{c})$ . Because of the maximality property  $\mu(L(\vec{c})) = \mu(L) + \bar{p}$ , each direct summand  $T_i$  of  $T$  satisfies  $\mu(L) \leq \mu(T_i) \leq \mu(L(\vec{c}))$  by Proposition 3.3.

By our assumption for  $L$  (resp.  $L(\vec{c})$ ) to correspond to a source (resp. a sink) of  $A$ , we may assume that  $L = T_1$  and  $L(\vec{c}) = T_n$ . Thus as in the proof of Theorem 1.4 we have that  $S_1 = T_1$  and  $S_n = \tau T_n[1] = T_1(\vec{c} + \vec{\omega})[1]$  and  $\text{rk}(S_n) = -\text{rk}(T_n) = -1$ , where  $S_1, \dots, S_n$  denote the simple  $A$ -modules corresponding to the indecomposable projective  $A$ -modules  $T_1, \dots, T_n$ . Invoking Lemma 3.1 we obtain

$$[T_1] - [T_n] + \sum_{i=2}^{n-1} \text{rk } S_i [T_i] = -\mathbf{w}.$$

Since  $T_1$  is a line bundle, we have equality  $[T_1(\vec{c})] = [T_1] + \mathbf{w}$  implying

$$\sum_{i=2}^{n-1} \text{rk } S_i [T_i] = 0.$$

Since the classes  $[T_1], \dots, [T_n]$  are linear independent in  $K_0(\text{coh } \mathbb{X})$ , each simple  $A$ -module  $S_i$  with  $i = 2, \dots, n-1$  has rank zero. Hence  $A$  has the maximal possible number of central simple modules. By Theorem 1.4 we then conclude that  $T = T_{\text{can}}$  up to a line bundle twist.  $\square$

**3.6. An addendum: tubular width.** For non-zero Euler characteristic the “distance”  $|\mu Y - \mu X|$  of a pair of objects is an invariant with respect to the autoequivalences of  $\text{D}^b(\text{coh } \mathbb{X})$ . This is no longer true for Euler characteristic zero where the “tubular distance” given by the absolute value of

$$\text{rk } X \text{ rk } Y (\mu Y - \mu X) = \begin{vmatrix} \text{rk } X & \text{rk } Y \\ \deg X & \deg Y \end{vmatrix} = \langle\langle X, Y \rangle\rangle$$

serves as a proper replacement. Here

$$\langle\langle X, Y \rangle\rangle = \sum_{j \in \mathbb{Z}_{\bar{p}}} \langle X, \tau^j Y \rangle, \quad \bar{p} = \text{lcm}(p_1, \dots, p_t)$$

is an average of the Euler form, and the above equality is Riemann-Roch’s theorem for a tubular weighted projective line  $\mathbb{X}$ , see [LM93]. For instance, each autoequivalence  $\sigma$  of  $\text{D}^b(\text{coh } \mathbb{X})$ , when applied to the canonical tilting bundle  $T_{\text{can}} = \bigoplus_{0 \leq \vec{x} \leq \vec{c}} \mathcal{O}(\vec{x})$ , yields tubular distance

$$\langle\langle \sigma \mathcal{O}, \sigma \mathcal{O}(\vec{c}) \rangle\rangle = \bar{p},$$

while  $|\mu(\sigma \mathcal{O}(\vec{c})) - \mu(\sigma \mathcal{O})| > 0$  can get arbitrarily small, see Theorem 5.7, (ii).

Assume  $\mathbb{X}$  is tubular, and  $T$  is a multiplicity-free tilting sheaf whose indecomposable summands  $T_1, \dots, T_n$  have monotonically increasing slope (with equality allowed). The question arises whether  $\langle\langle T_1, T_n \rangle\rangle = \bar{p}$  characterizes  $T_{\text{can}}$  up to autoequivalence of  $D^b(\text{coh } \mathbb{X})$ . We note (without proof) that this is indeed the case for tubular type  $(2, 2, 2, 2)$ , but fails for weight triples  $(3, 3, 3)$ ,  $(2, 4, 4)$  and  $(2, 3, 6)$ , as the following examples of Coxeter-Dynkin algebras of canonical type show. Note that these algebras are Schurian and that the relations are given by equation (6); moreover, they all have tubular width  $\langle\langle T_1, T_n \rangle\rangle = \bar{p}$ . Labels at vertices display the pair (degree, rank) as ‘fractions’. We remark further that a Coxeter-Dynkin algebra of type  $(2, 2, 2, 2)$  is isomorphic to the canonical algebra of the same type, so it does not qualify as a (counter)-example in the present context.

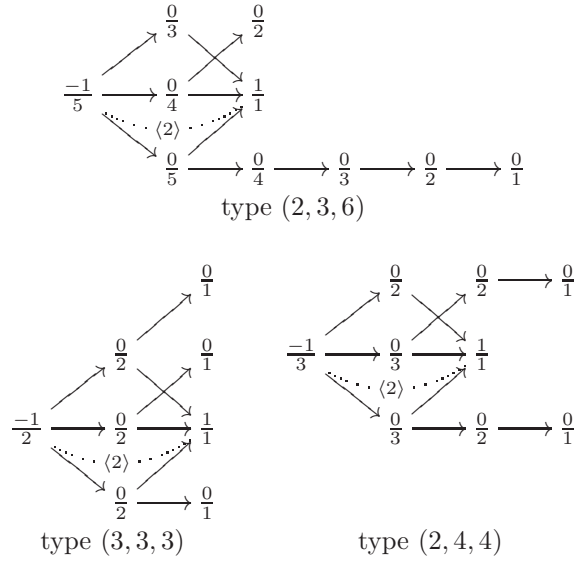
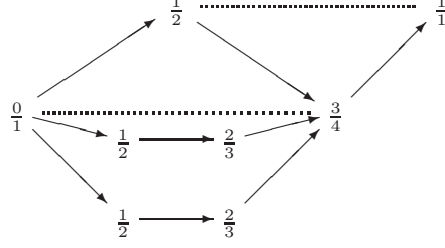


FIGURE 3.1. Tubular Coxeter-Dynkin algebras with  $\langle\langle T_1, T_n \rangle\rangle = \bar{p}$

#### 4. TWO INSTRUCTIVE EXAMPLES

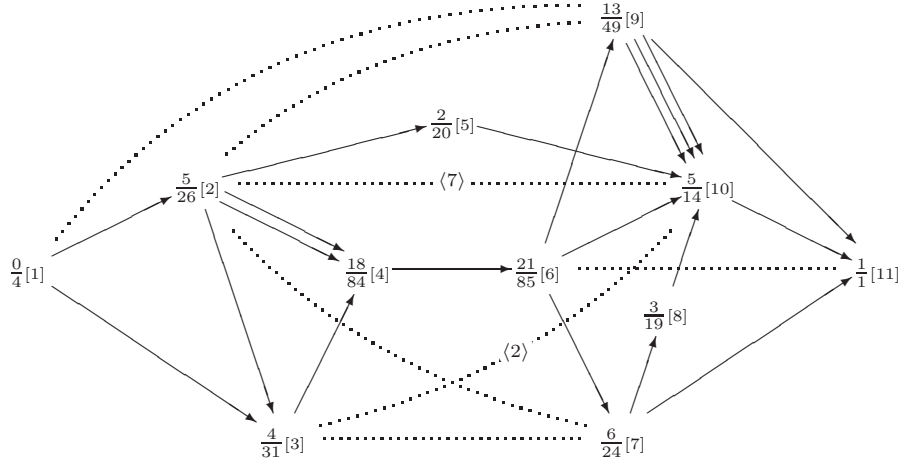
First we present two concealed canonical algebras  $A$  and  $B$ , one tubular, the other one wild, with interesting properties. We note that the quivers of  $A$  and  $B$  have a unique sink and a unique source.

**Example 4.1.** This example is the endomorphism ring of a tilting bundle  $T$  on a weighted projective line  $\mathbb{X}$  of tubular type  $(3, 3, 3)$ . Figure 4.1 shows a branch enlargement  $A$  of a canonical algebra of type  $(2, 3, 3)$ . The pair  $(\deg E, \text{rk } E)$  for each indecomposable summand  $E$  of  $T$  is displayed in the figure below, and also later, as the (unreduced) fraction degree/rank.

FIGURE 4.1. The algebra  $A$  of tubular weight type  $(3, 3, 3)$ .

We note that for each indecomposable summand  $E$  of  $T$  the pair  $(\deg E, \operatorname{rk} E)$  is coprime. By [LM93] this implies that  $E$  is quasi-simple in its tube which has (the maximal possible)  $\tau$ -period 3. This in turn implies that for any two indecomposable summands  $T'$  and  $T''$  of  $T$  there exists a self-equivalence  $u$  of the triangulated category  $D^b(\operatorname{coh} \mathbb{X})$  sending  $T'$  to  $T''$ . To phrase it differently, the tilting bundle  $T$  is homogeneous in  $D^b(\operatorname{coh} \mathbb{X})$ . But  $\operatorname{End}(T)$  is not a canonical algebra, implying by Theorem 1.1 that there is no self-equivalence  $v$  of  $D^b(\operatorname{coh} \mathbb{X})$  such that  $v(T)$  is a direct sum of line bundles.

**Example 4.2.** For weight type  $(2, 3, 7)$  where  $\chi_{\mathbb{X}} < 0$  there exists a tilting bundle  $T = \bigoplus_{i=1}^{11} T_i$  whose indecomposable summands  $T_i$  have rank and degree as shown in the next figure. Vertices are numbered [1] to [11], (unreduced) fractions  $\frac{d}{r}$  represent the pair  $(\text{degree}, \text{rank})$ . The quiver  $Q$  and the (minimal) numbers of relations for the endomorphism algebra  $B = \operatorname{End}(T)$  are displayed by Figure 4.2.

FIGURE 4.2. Algebra  $B$  of weight type  $(2, 3, 7)$ .

The most efficient way to construct tilting sheaves  $T$  as above is to apply Hübner reflections to the canonical configuration  $T_{\text{can}}$ , see Section 2.10. Here, one gets back from  $T$  to  $T_{\text{can}}$ , up to line bundle twist, by successive mutations in the following vertices 6, 4, 9, 8, 7, 8, 3, 5, 1, 2, 10, 5, 9, 10, 7, 3, 1, 10, 8, 2, 9, 4, 9, 7, 6, 4. Because we are

dealing with  $\leq 3$  weights, by [LM02] there exists, up to isomorphism, a unique endomorphism algebra  $B$  of a tilting bundle  $T$  with the given quiver and number of relations.

C.M. Ringel has collected in [Rin09] an impressive list of properties distinguishing canonical algebras within the class of tame concealed algebras, that is, the endomorphism rings of tilting bundles for a weighted projective line of Euler characteristic  $\chi_{\mathbb{X}} > 0$ . A number of these properties relies on an inspection of the Happel-Vossieck list classifying the tame concealed algebras [HV83].

In addition to the characterizing properties from Theorem 1.1, Theorem 1.3 and Theorem 1.4, Ringel states in [Rin09] that for a tame concealed algebra  $A$  (usually assumed to be not of type  $(p, q)$ ) each condition of the following list implies that  $A$  is canonical:

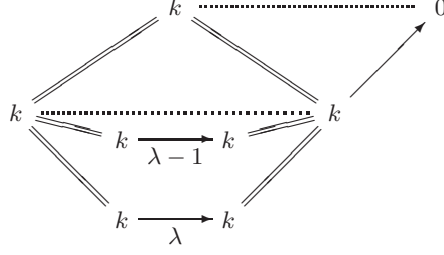
- (1)  $A$  has only one source and one sink.
- (2)  $A$  is not Schurian.
- (3) There exists a 2-Kronecker pair  $(X, Y)$  with  $X$  simple in  $\text{mod } A$ .
- (4) There exists a 2-Kronecker pair  $(X, Y)$  with  $Y$  simple in  $\text{mod } A$ .
- (5) There exists a one-parameter family of local modules.
- (6) There are local modules with self extensions.
- (7) There exists a one-parameter family of colocal modules.
- (8) There are colocal modules with self extensions.
- (9) There exists a projective indecomposable which is not thin.
- (10) There exists an injective indecomposable which is not thin.

Here, a pair  $(X, Y)$  is called a *2-Kronecker pair* if  $X, Y$  are exceptional, Hom-orthogonal, and with an extension space  $\text{Ext}^1(Y, X)$  of dimension two. An  $A$ -module  $X$  is called *local*, respectively *colocal*, if it has a unique maximal submodule (resp. a unique simple submodule). An  $A$ -module  $X \neq 0$  is called *thin* if for each indecomposable projective  $P$  the space  $\text{Hom}_A(P, X)$  has dimension at most one.

As shown by our next result, characterizations of canonical algebras within the class of tame concealed algebras have a tendency not to extend to the case of concealed canonical algebra in general, the major exceptions to this rule being those characterizations treated in Section 1.

**Proposition 4.3.** *None of the conditions (1)–(10) yields a characterization for canonical algebras in general.*

*Proof.* Both algebras,  $A$  and  $B$ , have only one source and only one sink and they are not Schurian, and they satisfy the conditions (1), (2), (9) and (10). We now show that  $B$  satisfies condition (3): Let  $X = S_3$  be the simple associated to vertex 3 and  $Y$  be the 2-dimensional indecomposable with top  $S_1$  and socle  $S_2$ . Then  $X$  and  $Y$  are exceptional objects which are Hom-orthogonal with  $\dim_k \text{Ext}^1(X, Y) = 2$ . For (4) repeat dualizing (3). For (5), (6), (7) and (8) we look at the  $A$ -modules given as representations in Figure 4.3. Note that the family is given by pairwise non-isomorphic indecomposables which are local and colocal and have self-extensions.

FIGURE 4.3. Distinct local, colocal and selfextending  $A$ -modules.

□

## 5. ALGEBRAS ANTIPODAL TO CANONICAL

Instead of maximality properties, as studied in Section 1, we now investigate the corresponding minimality properties. We start with a couple of properties of general interest.

**Useful generalities.**

**Proposition 5.1.** *Let  $\mathbb{X}$  be a weighted projective line,  $T$  be a tilting bundle and  $L$  be a line bundle on  $\mathbb{X}$ . Then either  $\text{Hom}(T, L) = 0$  or  $\text{Ext}^1(T, L) = 0$ .*

*Proof.* Assume that  $\text{Hom}(T, L) \neq 0$  and  $\text{Ext}^1(T, L) \neq 0$ . Invoking Serre duality, we obtain non-zero morphisms  $u : T \rightarrow L$  and  $v : L \rightarrow T(\vec{\omega})$ ; moreover  $v$  is a monomorphism since  $T$  is a vector bundle. Thus  $vu$  is non-zero in  $\text{Hom}(T, T(\vec{\omega})) = \text{DExt}^1(T, T) = 0$ , contradicting that  $T$  is tilting. □

The next result is due to T. Hübner [Hüb89], see also [LR06, Proposition 6.5]. It will play a central role when investigating minimality properties for positive Euler characteristic.

**Proposition 5.2** (Hübner). *Let  $\mathbb{X}$  be a weighted projective line with  $\chi_{\mathbb{X}} > 0$ . Then the direct sum of (a representative system of) the indecomposable vector bundles  $E$  with slope in the range  $0 \leq \mu E < |\delta(\vec{\omega})|$  is a tilting bundle  $T_{\text{her}}$  whose endomorphism ring  $A$  is hereditary. Moreover, the following holds:*

(i) *If  $t(\mathbb{X}) = 3$ , then each indecomposable summand  $E$  of  $T_{\text{her}}$  has slope 0 or  $|\delta(\vec{\omega})|/2$ . Correspondingly, each vertex in the quiver of  $A$  is a sink or a source.*

(ii) *If  $\mathbb{X}$  has weight type  $(p_1, p_2)$ ,  $1 \leq p_1 \leq p_2$  then  $T_{\text{her}}$  is the direct sum of all line bundles  $\mathcal{O}(\vec{x})$  with degree in the range  $0 \leq \delta(\vec{x}) \leq |\delta(\vec{\omega})| - 1 = \delta(\vec{x}_1) + \delta(\vec{x}_2) - 1$ . The quiver of  $A$  has bipartite orientation if and only if  $p_1 = p_2$ . □*

The next result is a reformulation of a result by Kerner and Skowroński [KS01, Theorem 3].

**Theorem 5.3** (Kerner-Skowroński). *Let  $\mathbb{X}$  be a weighted projective line of negative Euler characteristic. Further let  $m$  be a positive integer. Then there exists a tilting bundle  $T$  on  $\mathbb{X}$  such that for each indecomposable summand  $T_i$  of  $T$  and each simple*



**Minimal number of line bundle summands.** In this section we investigate the number of non-isomorphic line bundle summands of a tilting bundle  $T$  on  $\mathbb{X}$ . Note that the index  $[\mathbb{L} : \mathbb{Z}\vec{\omega}]$  equals the number of Auslander-Reiten orbits of line bundles. It is not difficult to see that  $\pm[\mathbb{L} : \mathbb{Z}\vec{\omega}] = p_1 \dots p_t \chi_{\mathbb{X}}$ , where the number on the right hand side is known as the *Gorenstein invariant* or *Gorenstein parameter* of the  $\mathbb{L}$ -graded coordinate algebra  $S = S(\underline{p}, \underline{\Delta})$  of  $\mathbb{X}$ . For positive Euler characteristic, we obtain the following values for  $[\mathbb{L} : \mathbb{Z}\vec{\omega}]$ :

weight type	$(p_1, p_2)$	$(2, 2, n)$	$(2, 3, 3)$	$(2, 4, 4)$	$(2, 3, 5)$
$[\mathbb{L} : \mathbb{Z}\omega]$	$p_1 + p_2$	4	3	2	1

(ii) Assume  $\chi_{\mathbb{X}} \leq 0$ . Then there exist a tilting bundle on  $\mathbb{X}$  without a line bundle summand.

A directed graph with nodes 1, 2, 3, 4, 5. Node 1 is on the left, connected to 2, which is connected to 3, which is connected to 4, which is connected to 5. Node 5 is connected to two nodes labeled 3, which are connected to node 4. Node 4 is connected to node 2.

(a)  $\text{Hom}(T, L) \neq 0$ , and    (b)  $\text{Hom}(T, L(\vec{\omega})) = 0$ .

We now assume  $\chi_{\mathbb{X}} = 0$ . By [LM00] there exists an autoequivalence  $\rho$  of  $D^b(\mathrm{coh}\, \mathbb{X})$  such that the induced map on pairs  $(\deg X, \mathrm{rk}\, X)^t$  is given by left multiplication with the matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Let  $T = \rho(T_{\text{can}}(\vec{u}))$  and  $A = \text{End}(T)$  where  $\vec{u}$  has degree one. Note that  $A$  is a canonical algebra; moreover the degree/rank distribution for the indecomposable summands of  $T_{\text{can}}(\vec{u})$  along the  $i$ th arm of the quiver of the canonical algebra  $A$  is given by

$$\frac{1}{1} \rightarrow \frac{1 + \bar{p}/p_i}{1} \rightarrow \frac{1 + 2\bar{p}/p_i}{1} \rightarrow \cdots \rightarrow \frac{1 + \bar{p}}{1}.$$

Applying  $\rho$  we obtain the corresponding degree/rank distribution for the indecomposables of the  $i$ th arm of  $T$  as

$$\frac{1}{2} \rightarrow \frac{1 + \bar{p}/p_i}{2 + \bar{p}/p_i} \rightarrow \frac{1 + 2\bar{p}/p_i}{2 + 2\bar{p}/p_i} \rightarrow \cdots \rightarrow \frac{1 + \bar{p}}{2 + \bar{p}}.$$

It follows that all ranks for the indecomposables in the  $i$ th arm of  $T$  have rank  $\geq 2$ , so that the claim follows.

Finally assume that  $\chi_{\mathbb{X}} < 0$ . Then the claim follows from Theorem 5.3 or Theorem 5.7.  $\square$

### Minimal number of bijections.

*Positive Euler characteristic.* Here the following cases arise:

- (a) Assume weight type  $(2, 3, p)$  with  $p = 3, 4, 5$  and consider the tilting bundle  $T_{\text{her}}$  from Proposition 5.2. For each arrow from the quiver  $Q$  of  $\text{End}(T)$  the source and sink have different rank. Accordingly there are no arrows  $u \rightarrow v$  inducing a bijection  $G_v \rightarrow G_u$  for the generic  $\text{End}(T)$ -module  $G$ .
- (b) Assume weight type  $(2, 2, p)$  with  $p \geq 2$ . Invoking [HV83] it follows that  $p - 2$  is the minimal number of arrows inducing a bijection. This number is attained for the tilting bundle  $T_{\text{her}}$  from Proposition 5.2.
- (c) Assume weight type  $(p_1, p_2)$  with  $1 \leq p_1 \leq p_2$ , and let  $T$  be any tilting bundle. Then the quiver  $Q$  of  $\text{End}(T)$  has  $n = p_1 + p_2$  vertices and also  $n$  arrows. Since all indecomposable summands of  $T$  have rank one, each arrow  $u \rightarrow v$  of  $Q$  induces a bijection  $G_v \rightarrow G_u$ .

*Euler characteristic zero.* We have shown in the proof of Proposition 5.4 that there exists a tilting bundle  $T$  whose endomorphism ring is the canonical algebra and such that the degree/rank distribution in the  $i$ th arm is given by

$$\frac{1}{2} \rightarrow \frac{1 + \bar{p}/p_i}{2 + \bar{p}/p_i} \rightarrow \frac{1 + 2\bar{p}/p_i}{2 + 2\bar{p}/p_i} \rightarrow \cdots \rightarrow \frac{1 + \bar{p}}{2 + \bar{p}}.$$

It follows that all ranks for the indecomposables in the  $i$ th arm are pairwise distinct. Hence no arrow  $u \rightarrow v$  induces a bijection  $G_v \rightarrow G_u$  for the  $T$ -distinguished generic  $\text{End}(T)$ -module  $G$ .

*Negative Euler characteristic.* For the minimal wild types  $(3, 3, 4)$ ,  $(2, 4, 5)$  and  $(2, 2, 2, 2, 2)$ , the degree/rank data for the tilting bundles  $T$  of Figure 5.1 show that no arrow  $u \rightarrow V$  of  $\text{End}(T)$  induces a bijection  $G_v \rightarrow G_u$ . For weight type  $(2, 3, 7)$  the same conclusion follows by inspection of Figure 4.2. Finally, for weight type  $(2, 2, 2, 3)$  we modify the example from Figure 5.1 by Hübner reflection in the sink [7] yielding an example with the wanted properties.



**Minimal number of central simple modules.** Let  $T$  be a tilting bundle on  $\mathbb{X}$  with endomorphism ring  $A$ . Recall that we identify  $\text{mod } A$  with a full subcategory of  $D^b(\text{coh } \mathbb{X})$  and call a simple  $A$ -module  $S$  *central simple* if  $S$  has rank zero, that is, belongs to  $\text{coh}_0 \mathbb{X}$ .

**Proposition 5.5.** *Let  $T$  be a tilting bundle on  $\mathbb{X}$  with endomorphism algebra  $A$  and non-isomorphic indecomposable summands  $T_1, \dots, T_n$ .*

(i) *Assume  $\chi_{\mathbb{X}} > 0$ . Then the following assertions hold.*

- (a) *If  $t(\mathbb{X}) = 3$  there exists a tilting bundle  $T_{\text{her}}$  with a hereditary endomorphism ring  $A$  and without central simple  $A$ -modules.*
- (b) *Assume weight type  $(p_1, p_2)$  with  $1 \leq p_1 \leq p_2$ . Then for each tilting bundle  $T$  the endomorphism ring  $A$  has at least  $p_2 - p_1$  central simple  $A$ -modules, and this bound is attained.*

(ii) *Assume  $\chi_{\mathbb{X}} = 0$ . Then there is a tilting bundle  $T$  such that  $A$  is canonical without central simple modules.*

(iii) *Assume  $\chi_{\mathbb{X}} < 0$ . Then there exist tilting bundles such that  $A$  has no central simple modules.*

*Proof.* Case (i)(a): The tilting bundle  $T_{\text{her}}$  from Proposition 5.2 has an endomorphism ring  $A$  whose quiver has bipartite orientation. Thus the simple  $A$ -module  $S_i$  attached to  $T_i$  equals  $T_i$  (resp.  $T_i[1]$ ) if  $i$  is a source (resp. a sink) of the quiver of  $A$ . In particular, each  $S_i$  has non-zero rank, and  $A$  has no central simple modules.

Case (i)(b): We refer to Lemma 5.6, proved below.

Case (ii): We thus consider the case where  $\mathbb{X}$  is tubular. By [LM00] there exists an autoequivalence  $\rho$  of  $D^b(\text{coh } \mathbb{X})$  such that the induced map on slopes is  $q \mapsto \frac{q}{1+q}$ . Let  $T = \rho T_{\text{can}}$  and  $A = \text{End}(T)$ . Then the simple  $A$ -modules all have slope 0, 1, or  $\frac{\bar{p}}{1+\bar{p}}$ . Hence none of these has rank zero.

Case (iii): Assume finally that  $\chi_{\mathbb{X}} < 0$ . By Theorem 5.3 there exist infinitely many tilting bundles  $T$  such that  $\text{End}(T)$  has no central simples.  $\square$

For illustration, we present explicit examples for the minimal wild weight types in Figure 5.1. For the three algebras of triple weight type the graphical information determines the algebras up to isomorphism, see [LM02]. For the two remaining weight types, the explicit relations are given below.

The following sequences of Hübner reflections, see Section 2.10, sequences transform the tilting bundles, depicted above, into  $T_{\text{can}}$ , up to line bundle twist:  $(2, 3, 7)$ :  $(10, 7, 11, 1, 2, 4, 7, 8, 9, 10, 11)$ ;  $(2, 4, 5)$ :  $(9, 6, 4, 5, 3, 2, 10, 1)$ ;  $(3, 3, 4)$ :  $(6, 5, 4, 3, 2, 9, 1)$ ;  $(2, 2, 2, 3)$ :  $(6, 4, 3, 2, 7)$ ;  $(2, 2, 2, 2, 2)$ :  $(7)$ .

Concerning  $(2, 2, 2, 3)$ , we impose the relations  $b_3 a_3 = b a_1$ ,  $b_4 a_4 = b a_2$ ,  $b_5 a_5 = b(a_2 - a_1)$ ,  $c(a_2 - \lambda a_1) = 0$  where  $\lambda$  is supposed to be different from 0, 1. Concerning  $(2, 2, 2, 2, 2)$ , we impose the relations  $b_1 a_i = \lambda_i b_1 a_1$  for  $i = 3, 4, 5$ ;  $b_2 a_i = b_2 a_j$  for  $i, j = 3, 4, 5$ ;  $b_j a_i = 0$  for  $j \neq 1, 2, i$ . We assume  $\lambda_3 = 1$  and  $\lambda_4 \neq \lambda_5$  to be different from 0, 1.

**Lemma 5.6.** *Assume  $\mathbb{X}$  of weight type  $(p_1, p_2)$ . Let  $T$  be a tilting bundle and  $Q$  the quiver of  $\text{End}(T)$ . Then the number  $\nu(T)$  of central simple  $A$ -modules equals*

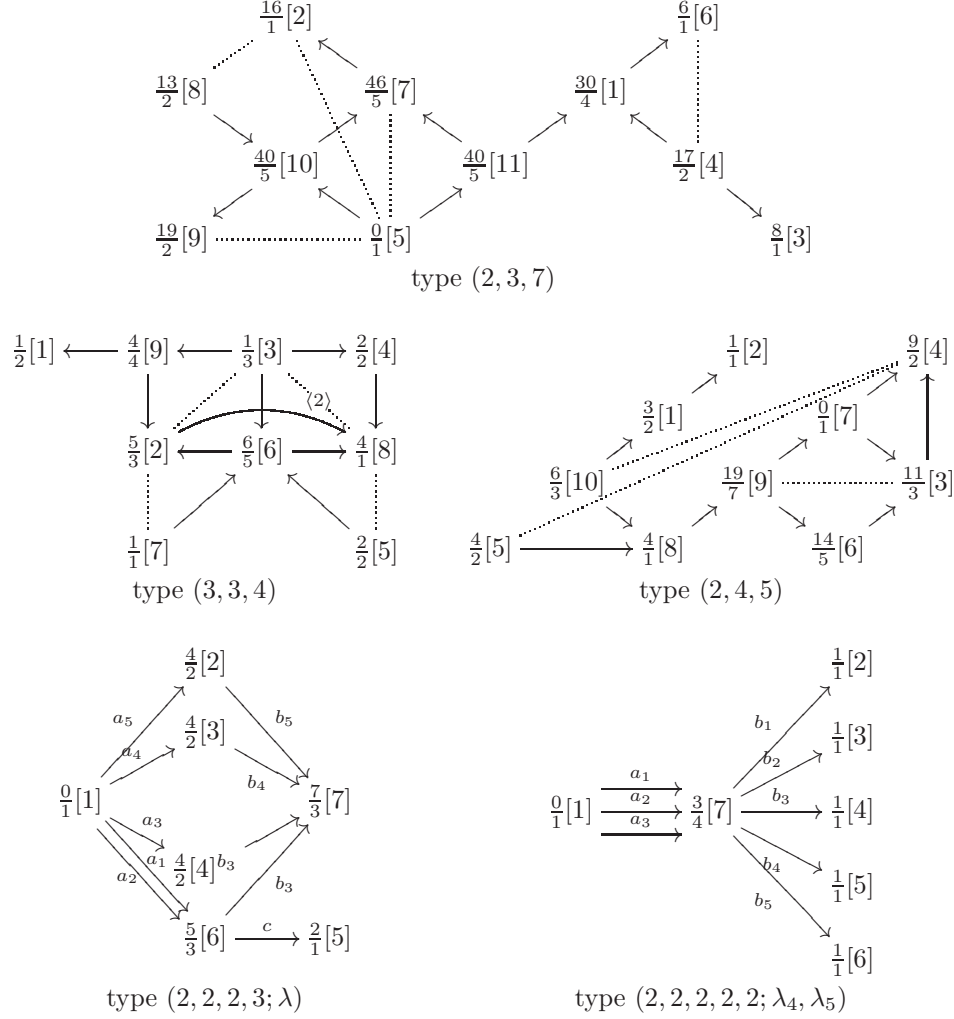
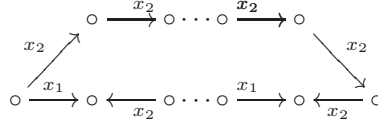


FIGURE 5.1. Minimal wild canonical type without central simple modules

the number of vertices of  $Q$  which are neither a sink nor a source. Always we have  $\nu(T) \geq |p_1 - p_2|$  with equality attained for the tilting object  $T$  given by the scheme



Assuming  $p_1 \leq p_2$ , the scheme contains  $p_1$  pairs  $\circ \xrightarrow{x_1} \circ \xleftarrow{x_2} \circ$  of arrows, labeled  $x_1$  and  $x_2$ , that are followed by  $p_2 - p_1$  arrows, labeled  $x_2$ , with arrows labeled  $x_2$  (resp.  $x_1$ ) having clockwise (resp. anticlockwise) orientation.

*Proof.* If  $i \in [1, n]$  is a source (resp. sink) of  $Q$ , then the simple  $A$ -module  $S_i$ , corresponding to  $T_i$ , has rank 1 (resp.  $-1$ ). Assume, conversely, that  $i$  is not a sink or a

source of  $Q$ , hence locally we have one of the two cases (a)  $i + 1 \xrightarrow{x_1} i \xrightarrow{x_1} i - 1$  or (b)  $i - 1 \xrightarrow{x_2} i \xrightarrow{x_2} i + 1$  where we say that  $i$  is an interior vertex. We claim that then  $S_i$  has rank zero. Assuming case (a), let  $U$  be the unique simple sheaf concentrated in the first exceptional point  $\lambda_1$  having the additional property  $\text{Hom}(T_i, U) \neq 0$  (and then  $\text{Hom}(T_i, U) = k$ ). We note that multiplication by  $x_2$  (resp. by  $x_1$ ) acts as the identity (resp. the zero map) on  $U$ . Since all the  $x_1$ -arrows of  $Q$  (we have  $p_1$  of them) have the same orientation, we conclude that  $\text{Hom}(T_j, U) = 0$  for each vertex  $j \neq i$ . Under our usual identification of modules and sheaves  $U$  thus equals the simple  $A$ -module  $S_i$ , which therefore has rank zero. This proves the first claim and further shows that  $\nu(T)$  equals the number of interior vertices  $i$  in the cyclic arrangement of labels  $x_1$  and  $x_2$ . It follows that  $\nu(T) \geq |p_1 - p_2|$ . The proof of the last claim is obvious.  $\square$

**Minimal width.** Tilting bundles of minimal width only exist for positive Euler characteristic as is shown in our next result.

**Theorem 5.7.** *Let  $\mathbb{X}$  be a weighted projective line.*

- (i) *Assume  $\chi_{\mathbb{X}} > 0$ . Then the minimal width for tilting bundles on  $\mathbb{X}$  equals  $|\delta(\vec{\omega})|/2$  for  $t(\mathbb{X}) = 3$  and  $\delta(\vec{x}_1) + \delta(\vec{x}_2)$  for  $t(\mathbb{X}) \leq 2$ .*
- (ii) *If  $\chi_{\mathbb{X}} \leq 0$  then there exists a sequence  $(T_n)$  of tilting bundles on  $\mathbb{X}$  such that the sequence  $(w(T_n))$  converges to zero and, moreover, each indecomposable summand  $E$  of  $T_n$  has rank  $\geq n$ .*

*Proof.* Concerning (i) we use Proposition 5.2 stating that the direct sum  $T$  of (a representative system of) the indecomposable vector bundles  $E$  of slope  $0 \leq \mu E < |\delta(\vec{\omega})|$  forms a tilting bundle. For  $t(\mathbb{X}) = 3$ , each indecomposable summand  $E$  of  $T$  actually has slope 0 or  $|\delta(\vec{\omega})|/2$ , showing that the width of  $T$  equals  $|\delta(\vec{\omega})|/2$ . For  $t(\mathbb{X}) \leq 2$ , each indecomposable vector bundle has rank one, such that  $T$  is the direct sum of all *line bundles*  $\mathcal{O}(\vec{x})$  with  $0 \leq \vec{x} \leq |\delta(\vec{\omega})| - 1$ . Thus in this case the width of  $T$  equals  $|\delta(\vec{\omega})| - 1 = \delta(\vec{x}_1) + \delta(\vec{x}_2) - 1$ .

In the tubular case assertion (ii) is covered by Proposition 5.9. For  $\chi_{\mathbb{X}} < 0$  the proof of (ii) is also given afterwards.  $\square$

We first assume that  $\mathbb{Y}$  is tubular, and collect some facts on the tubular mutations  $\sigma$  and  $\rho$  from Section 2.9. Let  $\mathbf{w} = [S_0]$  denote the class of any ordinary simple sheaf  $S_0$ . Further let  $\bar{p} = \bar{p}(\mathbb{Y})$  denote the largest weight of  $\mathbb{Y}$ . We first note that for each  $y$  from  $K_0(\text{coh } \mathbb{Y})$  we have

$$(16) \quad \sigma^{n\bar{p}}(y) = y + n \text{rk}(y) \mathbf{w},$$

a formula valid for any weight type. (This follows from the formula  $\bar{p}\vec{x} = \vec{c}$  if  $\vec{x}$  is one of the standard generators  $\vec{x}_i$  of  $\mathbb{L}$  of degree one.) By means of the conjugation formula (8),  $\rho^{-1} = (\rho^{-1}\sigma)^{-1}\sigma(\rho^{-1}\sigma)$ , we obtain a corresponding formula

$$(17) \quad \rho^{n\bar{p}}(y) = y + n \deg(y) \mathbf{z},$$

for the action of  $\rho$  on members  $y$  of  $K_0(\text{coh } \mathbb{Y})$ , where  $\mathbf{z}$  denotes the class of  $Z = \sigma^{-1}\rho(S_0)$ .

We call a bundle  $E$  on a weighted projective line  $\mathbb{X}$  *omnipresent on  $\mathbb{X}$*  if  $\text{Hom}(E, S) \neq 0$  for each simple sheaf  $S$ .

**Lemma 5.8.** *The indecomposable bundle  $Z = \sigma^{-1}\rho(S_0)$  is omnipresent on  $\mathbb{Y}$ .*

*Proof.* Let  $S$  denote any simple sheaf, say of degree  $d$ . Then  $\sigma S$  is again simple, having the same degree. Since  $\rho^{-1}$  acts on the degree-rank pair  $(d, 0)$  by right multiplication with  $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ , we obtain that  $(d, -d)$  is the degree-rank pair for  $\rho^{-1}\sigma S_0$ . It follows that

$$\dim \operatorname{Hom}(Z, S) = \langle Z, S \rangle = \langle S_0, \rho^{-1}\sigma S \rangle = -\operatorname{rk}(\rho^{-1}\sigma S) = d > 0,$$

as claimed.  $\square$

**Proposition 5.9.** *Assume that  $\mathbb{Y}$  is tubular. Let  $T = \bigoplus_{i=1}^m T_i$  be a tilting bundle on  $\mathbb{Y}$  whose indecomposable summands  $T_i$  all have strictly positive slope. For each integer  $n \geq 0$  we put*

$$T(n) = \rho^{n\bar{p}}T \quad \text{and} \quad T_i(n) = \rho^{n\bar{p}}T_i.$$

*Then each  $T(n)$  is a tilting bundle on  $\mathbb{Y}$  with endomorphism ring isomorphic to  $\operatorname{End}(T)$ . Moreover, the following holds for each  $i = 1, \dots, m$ :*

(a) *We have  $\operatorname{rk}(T_i(n)) > \bar{p}n$ , moreover the slope sequence  $(\mu T_i(n))_n$  converges to zero. In particular, the width sequence  $(w(T(n)))_n$  converges to zero.*

(b) *For each simple sheaf  $S$  on  $\mathbb{Y}$  we have  $\dim \operatorname{Hom}(T_i(n), S) \geq n$ .*

*Proof.* We put  $d_i = \deg T_i$ ,  $r_i = \operatorname{rk} T_i$  and use similarly  $d_i(n)$  and  $r_i(n)$  for the degree-rank data of  $T_i(n)$ . Then

$$(18) \quad (d_i(n), r_i(n)) = (d_i, r_i) \begin{pmatrix} 1 & n\bar{p} \\ 0 & 1 \end{pmatrix} = (d_i, r_i + n\bar{p}d_i).$$

By assumption we have  $r_i > 0$  and  $d_i \geq 1$ , hence  $\operatorname{rk}(T_i(n)) = r_i + n\bar{p}d_i > n\bar{p}$ . Moreover, the sequence of slopes

$$\mu(T_i(n)) = \frac{d_i}{r_i + n\bar{p}d_i}$$

obviously converges to zero. This proves assertion (a).

Concerning (b), we apply formula (17) to the class  $y = [T_i]$  and obtain  $[T_i(n)] = [T_i] + n d_i [Z]$ , hence  $\dim \operatorname{Hom}(T_i(n), S) = \langle [T_i(n)], [S] \rangle = \langle [T_i], [S] \rangle + n d_i \langle [Z], [S] \rangle \geq n$  where the inequality uses that  $d_i \geq 1$  and  $Z$  is omnipresent on  $\mathbb{Y}$  by Lemma 5.8.  $\square$

*Proof of Theorem 5.7, part (ii).* Next we assume that  $\chi_{\mathbb{X}} < 0$ . In several steps we are going to construct a sequence of tilting bundles  $T^*(n)$ ,  $n \geq 0$ , on  $\mathbb{X}$  satisfying the claims of Theorem 5.7.

*Step 1.* Let  $\bar{q} = (q_1, \dots, q_s)$  be the weight type of  $\mathbb{X}$ . After reordering we can write  $\bar{q} = \bar{p} + \bar{h}$  where  $\bar{p} = (p_1, \dots, p_t, 1, \dots, 1)$  is tubular and  $\bar{h} = (0, \dots, 0, h_r, \dots, h_s)$  has entries  $h_i \geq 1$  for  $i = r, \dots, s$ . For each of the (distinct) exceptional points  $x_i$  of  $\mathbb{X}$  with  $i = r, \dots, s$  we fix a linear branch  $B(i)$  of length  $h_i$  which is concentrated in  $x_i$ . Thus  $B(i) = \bigoplus_{j=1}^{h_i} U_j(i)$  where

$$(19) \quad B(i) : \quad U_{h_i}(i) \rightarrow U_{h_i-1}(i) \rightarrow \dots \rightarrow U_1(i)$$

consists of a chain of finite length sheaves concentrated in  $x_i$  such that each  $U_j(i)$  has length  $j$ , and hence  $U_1(i)$  is exceptional simple on  $\mathbb{X}$ . We call  $U_{h_i}(i)$  the *root* of  $B(i)$ . Then, putting  $B = B(r) \oplus B(r+1) \oplus \cdots \oplus B(s)$ , the right perpendicular category  $B^\perp$  of  $B$  in  $\text{coh } \mathbb{X}$  can be identified with the category of coherent sheaves on a weighted projective line  $\mathbb{Y}$  having tubular type  $\bar{p}$ , see [GL91]. Moreover, if  $T$  is a tilting bundle on  $\mathbb{Y}$ , then  $\bar{T} = T \oplus B$  is a tilting sheaf on  $\mathbb{X}$ , whose bundle part ‘lives on’  $\mathbb{Y}$ . For further details we refer to [LM96, Theorem 3.1 and Theorem 4.1].

*Step 2.* Keeping the notations of Proposition 5.9 we extend the tilting bundles  $T(n) = \rho^{n\bar{p}}T$  on  $\mathbb{Y}$  obtained by the preceding step to the tilting sheaf  $\bar{T}(n) = T(n) \oplus B$  on  $\mathbb{X}$ . Note that the embedding  $\text{coh } \mathbb{Y} \hookrightarrow \text{coh } \mathbb{X}$  preserves the rank but not the degree. In fact, the association  $\deg_{\mathbb{Y}} Y \mapsto \deg_{\mathbb{X}} Y$ , for  $Y$  in  $\text{coh } \mathbb{Y}$ , does not extend to a mapping  $K_0(\text{coh } \mathbb{Y}) \rightarrow K_0(\text{coh } \mathbb{X})$ , see [GL91, Section 9]. The following lemma will allow us to bypass this technical difficulty. We note that we continue to use the notations of Proposition 5.9.

**Lemma 5.10.** *For each  $i = 1, \dots, m$  the sequence  $(\mu_{\mathbb{X}}(T_i(n)))_n$  of slopes, formed in  $\text{coh } \mathbb{X}$ , converges to  $\frac{1}{\bar{p}} \deg_{\mathbb{X}} Z$ . Here  $Z = \sigma^{-1} \rho(S_0)$  is formed in  $\text{coh } \mathbb{Y}$  with  $S_0$  an ordinary simple sheaf on  $\mathbb{Y}$ , and  $\bar{p} = \text{lcm}(p_1, \dots, p_t)$ .*

*Proof.* We first note that formula (17) holds in  $K_0(\mathbb{Y})$ , hence in  $K_0(\mathbb{X})$ . Thus for each  $y \in K_0(\mathbb{Y})$  we have

$$(20) \quad \deg_{\mathbb{X}}(\rho^{n\bar{p}}(y)) = \deg_{\mathbb{X}}(y) + n \deg_{\mathbb{Y}}(y) \deg_{\mathbb{X}}(Z).$$

By formula (18) we obtain  $\text{rk}(\rho^{n\bar{p}}y) = \deg_{\mathbb{Y}}(y) + n \bar{p} \text{rk}(y)$ . Thus the slope sequence

$$\mu_{\mathbb{X}}(\rho^{n\bar{p}}y) = \frac{\deg_{\mathbb{X}}(y) + n \deg_{\mathbb{Y}}(y) \deg_{\mathbb{X}}(Z)}{\text{rk}(y) + n \bar{p} \deg_{\mathbb{Y}}(y)}$$

converges to  $\frac{1}{\bar{p}} \deg_{\mathbb{X}}(Z)$ .  $\square$

*Step 3.* By means of a sequence of Hübner reflections, we next transform  $\bar{T}(n)$  into a tilting bundle  $T^*(n)$  on  $\mathbb{X}$ . Let  $U_h$  be the root of a branch  $B = B(i)$ . Then  $U_h$  is a formal sink of  $\bar{T} = \bar{T}(n)$ , and reflection at  $U_h$  yields a new tilting sheaf  $\bar{T}/U_h \oplus U_h^*(n)$ , where  $U_h^*(n)$  is the kernel term of the reflection sequence

$$(21) \quad 0 \longrightarrow U_h^*(n) \longrightarrow \bigoplus_{j=1}^m T_j^{\kappa_j}(n) \longrightarrow U_h \longrightarrow 0,$$

compare formula (9). Since some exponent  $\kappa_j$  is non-zero, we see that  $U_h^*(n)$  is an exceptional vector bundle of rank  $r(n) = \sum_{j=0}^m \kappa_j \text{rk}(T_j(n)) > n$ . We next show that the slope sequence  $\mu_{\mathbb{X}}(U_h^*(n))$  also converges to  $\frac{1}{\bar{p}} \deg_{\mathbb{X}}(Z)$ . Clearly,

$$\mu_{\mathbb{X}}(U_h^*(n)) = \mu_{\mathbb{X}}\left(\bigoplus_{j=1}^m T_j^{\kappa_j}(n)\right) - \frac{\deg_{\mathbb{X}}(U_h)}{r(n)}.$$

Now the first summand  $\alpha(n)$  on the right hand side is a convex combination of the slopes  $\mu_{\mathbb{X}}(T_j(n))$ , and thus yields a sequence  $(\alpha(n))$  converging to  $\frac{1}{\bar{p}} \deg_{\mathbb{X}}(Z)$ , while the second summand  $\beta(n)$  yields a sequence  $(\beta(n))$  converging to zero. This proves the claim for this first step. We now continue reflecting roots of branches until all branches are exhausted, the resulting sequence of tilting bundles  $T^*(n)$  then satisfies all claims. This finishes the proof of Theorem 5.7 (ii).  $\square$

To construct explicit examples, the following result is useful.

**Proposition 5.11.** *Assume  $\mathbb{Y}$  is tubular, and  $T = \bigoplus_{0 \leq \vec{x} \leq \vec{c}} T_{\vec{x}}$  is a tilting bundle on  $\mathbb{Y}$  with  $\text{End}(T)$  canonical, accordingly with  $\text{Hom}(T_{\vec{x}}, T_{\vec{y}}) = \text{Hom}(\mathcal{O}(\vec{x}), \mathcal{O}(\vec{y}))$  for all  $0 \leq \vec{x}, \vec{y} \leq \vec{c}$ . We assume that the width  $w(T)$  of  $T$  is strictly less than  $\bar{p} = \bar{p}(\mathbb{Y})$ , the maximal possible one. Let further  $U$  be any sheaf of finite length. Then each morphism  $T_{\vec{x}} \rightarrow U$  factors through any non-zero morphism  $u : T_{\vec{x}} \rightarrow T_{\vec{c}}$ .*

We note that the assertion is wrong if  $T$  attains the maximal possible width  $\bar{p}$ . Indeed, by Theorem 1.5, we then may assume that  $T_{\vec{x}} = \mathcal{O}(\vec{x})$  holds for each  $\vec{x}$ . If  $S$  denotes the exceptional simple sheaf defined by exactness of  $0 \rightarrow \mathcal{O}(\vec{c} - \vec{x}_1) \rightarrow \mathcal{O}(\vec{c}) \rightarrow S \rightarrow 0$ , then  $\text{Hom}(\mathcal{O}(\vec{c} - \vec{x}_1), S(\vec{\omega})) = k$  but we further have  $\text{Hom}(\mathcal{O}(\vec{c}), S(\vec{\omega})) = 0$ .

*Proof of Proposition 5.11.* Since  $\text{End}(T)$  is canonical, there exists a self-equivalence  $\phi$  of  $\text{D}^b(\text{coh } \mathbb{Y})$  mapping  $\mathcal{O}(\vec{x})$  to  $T_{\vec{x}}$  for each  $0 \leq \vec{x} \leq \vec{c}$ . Any triangle  $\mu : T_{\vec{x}} \xrightarrow{u} T_{\vec{c}} \rightarrow V_{\vec{x}} \rightarrow 0$  is thus given as the image under  $\phi$  of a triangle represented by a short exact sequence  $\eta : 0 \rightarrow \mathcal{O}(\vec{x}) \xrightarrow{v} \mathcal{O}(\vec{c}) \rightarrow U_{\vec{x}} \rightarrow 0$  in  $\text{coh } \mathbb{X}$ . Having finite length, all the  $U_{\vec{x}}$  have the same slope. Hence all the  $V_{\vec{x}} = \phi(U_{\vec{x}})$  have the same slope  $q$ . If  $q = \infty$ , then all  $V_{\vec{x}}$  have rank zero, implying  $\text{rk } T_{\vec{x}} = \text{rk } T_{\vec{c}}$  for each  $0 \leq \vec{x} \leq \vec{c}$ . Then Theorem 1.1 implies that  $T = T_{\text{can}}$  up to a line bundle twist and hence  $w(T) = \bar{p}$ , contradicting our assumption on  $T$ .

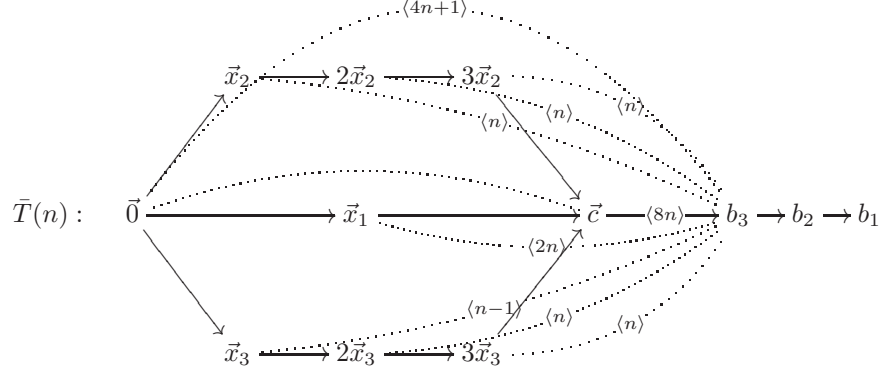
Thus each  $V_{\vec{x}}$  is a vector bundle and thus the triangle  $\mu$  yields an exact sequence  $\mu : 0 \rightarrow T_{\vec{x}} \xrightarrow{u} T_{\vec{c}} \rightarrow V_{\vec{x}} \rightarrow 0$  in  $\text{coh } \mathbb{Y}$  whose terms are vector bundles. Since  $\text{Ext}^1(-, U)$  vanishes on  $\text{vect } \mathbb{Y}$  for each  $U$  of finite length, the sequence

$$0 \rightarrow \text{Hom}(V_{\vec{x}}, U) \longrightarrow \text{Hom}(T_{\vec{c}}, U) \xrightarrow{-\circ u} \text{Hom}(T_{\vec{x}}, U) \rightarrow 0$$

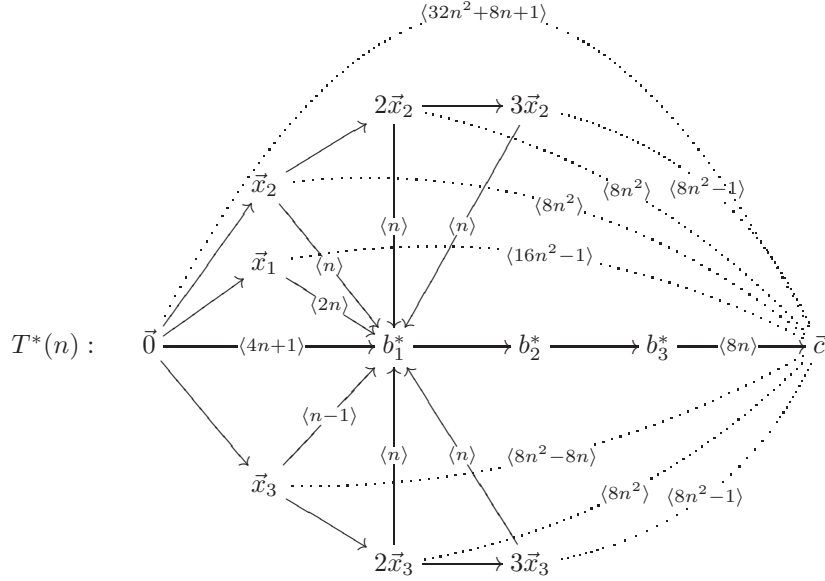
is exact, thus proving the claim.  $\square$

We are now constructing an explicit sequence of tilting bundles  $T^*(n)$  on the weighted projective line  $\mathbb{X}$  of weight type  $(2, 4, 7)$  illustrating the arguments of this section. We start with the tilting bundle  $T = T_{\text{can}}(\vec{c})$  on the tubular weighted projective line  $\mathbb{Y}$  of weight type  $(2, 4, 4)$ , and form the sequence of  $T(n)$  of tilting bundles of Proposition 5.9. Fixing a branch  $B : U_3 \twoheadrightarrow U_2 \twoheadrightarrow U_1$  of length 3 concentrated in the third exceptional point of  $\mathbb{X}$  we identify  $\text{coh } \mathbb{Y}$  with the perpendicular subcategory  $B^\perp$  in  $\text{coh } \mathbb{X}$ , and then enlarge  $T(n)$  to the tilting sheaf  $\bar{T}(n) = T(n) \oplus B$ . The endomorphism ring of  $\bar{T}(n)$  is then given by the following

quiver with relations.



This uses Proposition 5.11. Applying Hübner reflections in the vertices  $b_3, b_2, b_1$  in this order, we finally obtain a sequence of tilting bundles on  $\mathbb{X}$  whose endomorphism rings are given as follows.



The degree, rank and slope data for the tilting bundles  $T^*(n)$  are collected in the following table.

$\vec{0}$	$\vec{x}_1$	$\vec{x}_2$	$2\vec{x}_2$	$3\vec{x}_2$	$\vec{x}_3$
$\frac{8n+24}{16n+1}$	$\frac{12n+38}{24n+1}$	$\frac{10n+31}{20n+1}$	$\frac{12n+38}{24n+1}$	$\frac{14n+45}{28n+1}$	$\frac{10n+31}{20n+1}$
$2\vec{x}_3$	$3\vec{x}_3$	$\vec{c}$	$b_1^*$	$b_2^*$	$b_3^*$
$\frac{12n+44}{24n+1}$	$\frac{14n+48}{28n+1}$	$\frac{16n+52}{32n+1}$	$\frac{128n^2+416n-12}{236n^2+8n}$	$\frac{128n^2+416n-8}{256n^2+8n}$	$\frac{128n^2+416n-4}{256n^2+8n}$

We observe that all the slope sequences converge to  $\frac{1}{2}$ . Further, all relations for  $T^*(n)$  end in the vertex  $\vec{c}$ . This can be rephrased as follows: Let  $Q(n)$  be the wild quiver, obtained from the quiver of  $T^*(n)$  by removing the last vertex  $\vec{c}$ . Then  $\text{End}(T^*(n))$  is obtained as a one-point extension of the path algebra  $kQ(n)$  of  $Q(n)$ .



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